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More on $\mathcal{N}=1$ matrix model curve for arbitrary N

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Abstract: Using both the matrix model prescription and the strong-coupling approach, we describe the intersections of $n=0$ and $n=1$ non-degenerated branches for quartic (polynomial of adjoint matter) tree-level superpotential in $\mathcal{N}=1$ supersymmetric SO(N)/USp($2N$) gauge theories with massless flavors. We also apply the method to the degenerated branch. The general matrix model curve on the two cases we obtain is valid for arbitrary N and extends the previous work from strong-coupling approach. For SO(N) gauge theory with equal massive flavors, we also obtain the matrix model curve on the degenerated branch for arbitrary N. Finally we discuss on the intersections of $n=0$ and $n=1$ non-degenerated branches for equal massive flavors.

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Contents

1. Introduction

The exact quantum effective superpotential for the glueball field was proposed by Dijkgraaf and Vafa [1, 2, 3] using a zero-dimensional matrix model. Extremization of the effective glueball superpotential has led to the quantum vacua of the supersymmetric gauge theory. For $\mathcal{N} = 1$ supersymmetric $\mathrm{U}(N)$ gauge theory with the adjoint matter Φ, the gauge group $\mathrm{U}(N)$ breaks into $\prod_{i=1}^{n} \mathrm{U}(N_i)$ for some n where $N = \sum_{i=1}^{n} N_i$. At low energies, the effective theory becomes $\mathcal{N} = 1$ gauge theory with gauge group $\mathrm{U}(1)^n$. The low energy dynamics of this gauge theory along the line of [1, 2, 3] have been studied in [4]–[10].

Although the $\mathcal{N} = 2$ factorization problem in the strong-coupling approach [11, 12] has been solved for small values of N explicitly [5], the difficulty of the solving the general (arbitrary N) factorization problem occurs. In [13], a matrix model curve for $\mathrm{U}(N)$ gauge theory with cubic tree-level superpotential where the gauge symmetry breaks into $\mathrm{U}(N) \to \mathrm{U}(N_1) \times \mathrm{U}(N_2)$ with $N = N_1 + N_2$ was studied. One of the lessons is that the description of minimization of glueball superpotential is useful to obtain a matrix model curve at $n = 1$ and $n = 2$ singularity[1] with *arbitrary* N for $\mathrm{U}(N)$ gauge theory. On the singularity, the two-branch cuts of the matrix model curve meet each other. In general, a matrix model curve

[1]For $\mathrm{U}(N)$ gauge theory, the tree-level superpotential is written as $W(\Phi) = \sum_{k=0}^{n} \frac{g_k}{k+1} \mathrm{Tr}\Phi^{k+1}$. Thus, the cubic case can be described by the value $n = 2$. On the other hand, for $\mathrm{SO}(N)/\mathrm{USp}(2N)$ gauge theories, the tree-level superpotential can be written as $W(\Phi) = \sum_{k=1}^{n+1} \frac{g_{2k}}{2k} \mathrm{Tr}\Phi^{2k}$. The quartic tree-level superpotential, which we will deal with mainly, is represented by the value $n = 1$.

can be represented as $y_m^2 = W_2'(x)^2 + f_1(x) = \prod_{i=1}^4 (x - x_i)$ in terms of either a parameter of superpotential and fields or the roots of y_m. The period integrals on the Riemann surface can be written by the elliptic integrals. The glueball equations of motion allowed us to solve for the roots x_i in terms of parameters N_1, N_2 and a parameter of superpotential. It was not transparent to obtain the four different general roots x_i. However at the singularity, since the root x_2 approaches the root x_3 (reducing of the roots of y_m), by tuning the parameter of superpotential to special values depending on N_1 and N_2, it was possible to obtain the roots and parameter of superpotential for general (N_1, N_2). This overcomes the difficulty in the strong-coupling approach (factorization problem) when N is large because a matrix model curve is valid for *arbitrary* N.

The extension [5] to the $\mathcal{N} = 1$ supersymmetric gauge theories with the gauge groups $\mathrm{SO}(N)/\mathrm{USp}(2N)$ was obtained in [14]. The factorization problem gave an explicit constructions for the matrix model curve for small values of N. In [15], the description for $\mathrm{U}(N)$ gauge theory was generalized to the $\mathrm{SO}(N)/\mathrm{USp}(2N)$ gauge theories with quartic tree-level superpotential ($n = 1$) by following the method of [13]. In particular, a new result for degenerated branch[2] was obtained besides a simple generalization of the results in [13] to these gauge groups. For the degenerated case, the number of roots of y_m is less than the number of roots on the non-degenerated case. The matrix model curve for the degenerated case[3] can be written as

$$y_{m,d}^2 = \left(\frac{W_3'(x)}{x}\right)^2 + 4F \equiv \left(x^2 + x_0^2\right)\left(x^2 + x_1^2\right)$$

in terms of either a parameter of superpotential and field or the roots of y_m. Using a matrix model curve for non-degenerated case, $y_m^2 = \prod_{i=0}^2 \left(x^2 + x_i^2\right)$, the degenerated curve can be related to $y_m^2 = x^2 y_{m,d}^2$ implying that one of the roots in the non-degenerated case vanishes. This situation is quite similar to the one studied in [13]. Therefore, the claim in [15] was that the glueball approach is suitable for computation of a general matrix model curve on the degenerated case which depends on the field F, N_0 and N_1 for general breaking pattern $\mathrm{SO}(N) \to \mathrm{SO}(N_0) \times \mathrm{U}(N_1)$ with $N = N_0 + 2N_1$. The N can be any *generic* value. We should not restrict our discussion to a special point on a matrix model curve but it covers the whole degenerated branch. Thus, as studied in [14] there exist some smooth interpolations between the vacua with different breaking patterns like as Coulomb branch in [5].

[2]For $\mathrm{SO}(N)/\mathrm{USp}(2N)$ gauge theories, since a matrix model has $\mathbf{Z}_2$ identification the curve can be written in terms of a function of x^2. One can observe that the position of the branch cut on the fixed point cannot move freely by changing a scale Λ. If there are no branch cuts on the fixed point, called degenerated case, all the branch cuts move freely on the Riemann surface as illustrated in figure 4 in [5]. Therefore, the physics of these two cases are different from each other. Thus, we have to study these two cases separately. However, for $\mathrm{U}(N)$ gauge theory since there is no fixed point, we do not need to study them separately. This is one of the reasons why we do not study $\mathrm{U}(N)$ gauge theory with flavors in this paper. It would be interesting to study $\mathrm{U}(N)$ theory flavors in the future.

[3]For the cases of $N_f = N - 3$ for $\mathrm{SO}(N)$ and $N_f = 2N + 1$ for $\mathrm{USp}(2N)$ gauge theories, the extra piece proportional to x^2 should be added. See sections 3 and 5.

JHEP02(2004)009

It is natural and interesting to describe the glueball approach by adding the massive or massless flavors into the pure $\mathrm{SO}(N)/\,\mathrm{USp}(2N)$ gauge theories. Recall that the Seiberg-Witten (SW) curves [16, 17, 18] for $\mathrm{SO}(N)/\,\mathrm{USp}(2N)$ gauge theories with flavors are characterized by

$$\begin{aligned} y^2_{\mathrm{SO}(N)} &= P^2_{2[N/2]}(x) - 4\Lambda^{2(N-2-N_f)} x^{2(1+\epsilon)} \prod_{f=1}^{N_f} \left(x^2 - m_f^2\right), \\ x^2 y^2_{\mathrm{USp}(2N)} &= \left(x^2 P_{2N}(x) + 2\Lambda^{2N+2-N_f} \prod_{f=1}^{N_f} m_f\right)^2 - 4(-1)^{N_f} \Lambda^{2(2N+2-N_f)} \prod_{f=1}^{N_f} \left(x^2 - m_f^2\right) \end{aligned}$$

where $\epsilon = 0$ for N odd and $\epsilon = 1$ for N even. When the number of flavors N_f becomes zero in the power of Λ and the expressions containing the product for the mass m_f become 1, then the above formula reduces to the SW curve of pure gauge theory. Compared with the pure case, the factorization problem in the strong-coupling approach with flavors becomes more complicated due to the presence of flavor-dependent part. In the strong-coupling approach the $\mathcal{N} = 1$ matrix model curve is described by the single root part in the SW curve. One of our aims is to observe the singularity on the matrix model curve (i.e.,double roots part) for *arbitrary* N in our gauge theories [19, 20] from the point of view of glueball approach. We expect to have a general matrix model curve for massless flavors without any difficulty because the analysis from the strong-coupling approach implies that the SW curve looks like the curve for pure gauge theories with different power behavior of x above. However, for massive case (even equal mass for flavors), since we have an extra mass parameter in our factorization problem, most of the matrix model curves for given N and N_f (the number r in the r-th vacua) in our gauge theories do not have an extra double root and therefore, they do not exist at the singularity we are interested in. Instead, when they have an extra overall factor x^2 in the matrix model curve (i.e., degenerated case), there is a chance to obtain the general matrix model curve because the number of roots is reduced and this fact will make easier to compute the equation of motion for glueball field.

In this paper, we deal with two topics:

- One is $n = 0$ and $n = 1$ singularity on the non-degenerated case. This is a simple generalization of the results in [13] to the gauge theories with flavors in the vector representation for $\mathrm{SO}(N)$ gauge theory or fundamental representation for $\mathrm{USp}(2N)$ gauge theory.

- The other is a study for a general matrix model curve for the gauge theories on the whole degenerated branch.

The effective superpotential for the gauge theories with some flavors was already discussed in [6, 21, 22, 23, 24]. The contributions of the flavors to the effective superpotential, $\mathcal{F}_{\text{flavor}}$, can be expressed by the integral of matrix model curve (2.1). In [25, 19, 20], the mass of flavors possesses the same value and there exists one constraint, $W_3'(\pm m_f) = 0$.

For the $\mathrm{SO}(N)/\,\mathrm{USp}(2N)$ cases, since the superpotential $W_4(x)$ is an even polynomial in x, the relation $W_3'(x) = 0$ always has a zero $x = 0$ as a root. In other words, for massless case, the above constraint is automatically satisfied. Therefore, the massless case and the massive case are quite different from each other (One cannot obtain the massless case as we take the zero limit of the flavor mass in the massive case). As discussed in [19, 20], the massless flavors are charged under the factor $\mathrm{SO}(N_0)$, while the massive flavors are charged under the factor $\mathrm{U}(N_1)$ in the breaking pattern $\mathrm{SO}(N) \to \mathrm{SO}(N_0) \times \mathrm{U}(N_1)$.[4]

The organization of this paper is as follows. In section 2, we study the matrix model curve at $n = 0$ and $n = 1$ singularity for $\mathrm{SO}(N)$ gauge theory with massless flavors. This is the simplest example, because the contributions coming from the flavors can be written in terms of the dual period integral Π_0. Therefore, using previous results obtained in [15] for the pure gauge theory, we obtain a general matrix model curve (2.2) at the singularity for the massless flavor case by minimizing the effective superpotential with respect to the glueball field. To demonstrate on the validity of general matrix model curve, we deal with some explicit examples for $\mathrm{SO}(N)$ gauge theories where $N = 4, 5, 6, 7, 8$ or 9 with $N_f(\leq N-3)$ flavors. We have checked the precise values N_0 and N_1 explicitly by performing the period of $T(x)$ (characterized by a characteristic function) over the compact cycles, for some particular examples. The addition map is used.

In section 3, we study a general matrix model curve on degenerated branch for $\mathrm{SO}(N)$ gauge theory with massless flavors. Contrary to the results in section 2, we obtain the matrix model curve characterized by a single field. There are two classes of matrix model curves, (3.1) and (3.2), depending on whether the quantity $(N_0 - N_f - 2)$ is equal to zero or not. After we obtain the general matrix model curve for arbitrary N for $\mathrm{SO}(N)$ gauge theory, we explicitly check the equivalence between two approaches (glueball approach and the factorization problem of SW curve). The subtlety for the matrix model curve is discussed when the number of flavor N_f reaches the maximum value $(N-3)$ in the asymptotic region of the theory. We predict the matrix model curve for the degenerated $\mathrm{SO}(N)$ gauge theories with $N_f(\leq N-3)$ massless flavors where $N \geq 7$ and $\mathrm{SO}(N) \to \widehat{\mathrm{SO}(N_0)} \times \mathrm{U}(N_1), N_0 \neq 0$.

In sections 4 and 5, we extend the studies given in sections 2 and 3 for $\mathrm{SO}(N)$ gauge theories to the $\mathrm{USp}(2N)$ gauge theory with massless flavors. The procedures are similar to the $\mathrm{SO}(N)$ gauge theory. However, there exists one difference: the matrix model curves for the two gauge theories can be represented as $y_m^2 = x^2\left(x^2 \pm m\right)^2 + f_2(x)$ where the plus sign corresponds to the $\mathrm{SO}(N)$ gauge theory and the minus sign corresponds to the $\mathrm{USp}(2N)$ gauge theory. Therefore, by replacing $(N-2) \to (2N+2)$ and $x_i^2 \to -x_i^2$, we obtain the general matrix model curves (4.1), (5.1), and (5.2) for $\mathrm{USp}(2N)$ theory from $\mathrm{SO}(N)$ theory. When the number of flavors N_f reaches its maximum value $(2N+1)$ in a theory, the analysis for the curve in degenerated case needs to have an extra term due to the flavor dependent term. We predict the matrix model curve for the degenerated $\mathrm{USp}(2N)$ gauge theories with $N_f(\leq 2N+1)$ massless flavors where $N \geq 4$ and $\mathrm{USp}(2N) \to \widehat{\mathrm{USp}(2N_0)} \times \mathrm{U}(N_1), N_0 \neq 0$.

[4]Then, we use $\widehat{\mathrm{SO}(N_0)}$ or $\widehat{\mathrm{U}(N_1)}$ to represent the flavors which are charged under the corresponding gauge groups. Similar arguments hold for $\mathrm{USp}(2N)$ gauge theory.

In section 6, we study the degenerated $\mathrm{SO}(N)$ gauge theories with massive flavors. Contrary to the massless cases, the constraint for the mass of flavors, $W_3'(\pm m_f) = 0$, is *not* automatically satisfied. However, this condition provides that the contribution arising from flavors can be written in terms of a dual period Π_1. By using this fact and performing the equation of motion for glueball field, we obtain a general matrix curve for $\mathrm{SO}(N)$ theory. Moreover, we discuss r-Higgs branch discussed in [26, 27, 28, 19, 20, 6] in the context of glueball approach. To obtain a general matrix model curve for this branch we follow the discussion given in [6] about contour of integrals under the change of mass parameter and finally obtain the effective superpotential for the branch. Minimizing the effective superpotential we derive the matrix model curve for the theories (6.3), (6.4), (6.7), and (6.8). After the general discussions, we deal with some explicit examples, $\mathrm{SO}(N)$ with $N = 4, 5, 6$ and check the explicit agreement between two approaches. We predict the solutions for the degenerated $\mathrm{SO}(N)$ gauge theories with $N_f(\leq N-3)$ massive flavors where $N \geq 7$ and $\mathrm{SO}(N) \to \mathrm{SO}(N_0) \times \widehat{\mathrm{U}(N_1)}, N_0 \neq 0$.[5]

In section 7, we give some comments on the $n = 0$ and $n = 1$ singularities for massive $\mathrm{SO}(N)/\,\mathrm{USp}(2N)$ cases. We do not obtain the general matrix model curves because, in general, the $\mathcal{F}_{\text{flavor}}$ cannot be written as a dual period Π_1. Therefore, the matrix model curve cannot be represented by any simple formulation. However, when the condition, $c < 0$ where c is defined in (7.1) or (7.2), should be satisfied, the flavor-dependent part can be written in terms of Π_1. Thus, in this particular case, we obtain a matrix model curve for arbitrary N.

There exist many related works from the different matter representations for various gauge theories [29]-[37].

2. The $n = 0$ and $n = 1$ singularity: $\mathrm{SO}(N)$ gauge theory with massless flavors

Let us consider $\mathrm{SO}(N)$ gauge theory with quartic tree-level superpotential ($n = 1$) and N_f flavors. We are interested in the intersections of the $n = 0$ and $n = 1$ branches. At these special points, a vacuum with unbroken gauge group $\widehat{\mathrm{SO}(N_0)} \times \mathrm{U}(N_1)$ meets a vacuum with unbroken gauge group $\mathrm{SO}(N)$ with $N = N_0 + 2N_1$. Since we are considering an asymptotic free theory, we restrict the number of flavors N_f to satisfy the condition $N_f < N - 2$. As already clarified in [19], the relation between $W_3'(x) = x\left(x^2 + m\right)$ and $F_6(x)$ corresponding to an $\mathcal{N} = 1$ matrix model curve is characterized by

$$
\begin{aligned}
y_m^2 &= F_6(x) = W_3'(x)^2 + f_2(x)\,, && N_f < N-3\,, \\
y_m^2 &= F_6(x) = W_3'(x)^2 + f_2(x) - 4x^4\Lambda^2\,, && N_f = N-3
\end{aligned}
$$

where $f_2(x) = f_2 x^2 + f_0$. These relations are valid for any $\mathrm{SO}(N)$ gauge theories where N is even or odd. All masses of the flavors vanish. The Λ is an $\mathcal{N} = 2$ strong-coupling scale. The fluctuating fields f_2 and f_0 relate to the glueball fields S_1 and S_0 respectively.

[5] As one can see in [20], there is no degenerated branch for $\mathrm{USp}(2N)$ theory. Therefore, we do not need to study them here.

The effective superpotential for the SO(N) gauge theory with N_f flavors is given by [22, 21, 23, 24]

$$W_{\rm eff} = 2\pi i\left[(N_0-2)\Pi_0 + 2N_1\Pi_1\right] - 2(N-2-N_f)S\log\left(\frac{\Lambda}{\Lambda_0}\right) - \mathcal{F}_{\rm flavor}\,,$$

where the contributions from the flavors are given by the integral for y_m over x:

$$\mathcal{F}_{\rm flavor} = \sum_{f=1}^{N_f}\int_{\pm m_f}^{\Lambda_0} y_m dx\,. \tag{2.1}$$

The log-divergent term was renormalized by the bare Yang-Mills coupling and the cut-off Λ_0 was replaced by the physical scale Λ.[6] Note that there exists a relation $S = S_0 + 2S_1$. The periods S_i and dual periods Π_i of holomorphic 3-form for the deformed geometry were written as the integrals over the x-plane [15, 38, 39]. The m_f is the mass of the f-th flavor and we have chosen the location of holomorphic 2-cycles at $x = \pm m_f$. Around these points, D5-branes are wrapping on holomorphic 2-cycles, providing a source for a single unit of RR flux at $x = \pm m_f$. Thus the integral of 3-form RR flux around $x = \pm m_f$ is equal to 1. The full effective superpotential can be generated by adding this contribution to the effective superpotential for pure gauge theory.

First, let us consider the simple case in which all masses of flavors are zero in this section. Later, in sections 6 and 7, we will deal with the massive cases. Since the contribution from the flavors, $\mathcal{F}_{\rm flavor}$, becomes $2\pi i N_f\Pi_0$, we can use the results for pure SO(N) gauge theory studied in [15] without much efforts. We only have to replace the quantity (N_0-2) appeared in the matrix model curve in [15] with (N_0-N_f-2). Therefore, by combining the flavor-dependent part with the dual period Π_0 term, the effective glueball superpotential for massless flavors is

$$W_{\rm eff} = 2\pi i\left[(N_0-2-N_f)\Pi_0 + 2N_1\Pi_1\right] - 2(N-2-N_f)S\log\left(\frac{\Lambda}{\Lambda_0}\right).$$

The periods of $T(x)$ defined as the periods of the 1-form at the $n=0$ and $n=1$ singularity are given similarly and together with different power behavior of x and Λ, due to the presence of the flavors, the $T(x)$ also takes the form as a complete differential. With the above replacement in mind, we immediately obtain the matrix model curve near the singularity on the non-degenerated case from the glueball equations of motion:

$$y_m^2 = x^2\left(x^2+m\right)^2 + \langle f_2\rangle x^2 + \langle f_0\rangle = \left[x^2 + 2\eta\Lambda^2\left(1+c\right)\right]^2\left(x^2+4\eta\Lambda^2\right),$$

where the fields, parameter of a superpotential, and the glueball field are given by

$$\begin{aligned}
\langle f_2\rangle &= 4\eta^2\Lambda^4(1+2c)\,, \qquad & \langle f_0\rangle &= 16\eta^3\Lambda^6(1+c)^2\,,\\
m &= 2\eta\Lambda^2(2+c)\,, \qquad & \langle S\rangle &= -\eta^2\Lambda^4(1+2c)\,, \qquad c\equiv\cos\left(\frac{2\pi N_1}{N-N_f-2}\right).
\end{aligned} \tag{2.2}$$

[6] For the tree-level superpotential of degree $2(n+1)$, the effective superpotential for the SO(N) gauge theory with N_f flavors is $W_{\rm eff} = 2\pi i[(N_0-2)\Pi_0 + \sum_{i=1}^n 2N_i\Pi_i] - 2(N-2-N_f)(S_0+\sum_{i=1}^n 2S_i)\log(\Lambda/\Lambda_0) - \mathcal{F}_{\rm flavor}$. For quartic case ($n=1$), this will lead to the above effective superpotential.

Here η is the $(N-N_f-2)$-th root of unity for $(N-N_f-2)$ even and the $(N-N_f-2)$-th root of minus unity for $(N-N_f-2)$ odd. That is,

$$\eta^{N-N_f-2} = 1\,, \qquad \text{for } (N-N_f-2) \text{ even},$$
$$\eta^{N-N_f-2} = -1\,, \qquad \text{for } (N-N_f-2) \text{ odd}.$$

This matrix model curve with massless flavors is exactly the same as pure case with the replacement $(N-2) \to (N-N_f-2)$.[7] Note that the above solutions have the special combination $(N-N_f-2)$ which reflects the addition map discussed in [19], relating the Chebyshev branches (or the Special branches) of two different gauge groups. Namely, we have seen that the Chebyshev branch of SO(N) with N_f massless flavors can be reduced to the Chebyshev branch of SO(N') with N'_f massless flavors when there exists a relation: $N-N_f-2 = N'-N'_f-2$.

Let us demonstrate these general features (2.2) explicitly by comparing them with the results from strong-coupling approach obtained in [19] already. Let us consider SO(N) gauge theories with $N_f(\leq N-3)$ flavors where $N = 4,5,6,7,8$ or 9.

SO(4) with $N_f = 1$. In [19], the factorization problem resulted in the matrix model curve $\widehat{y_m}^2$ which contains an overall factor x^2.[8] Therefore, there exists no solution for non-degenerated case. We will come to this for degenerated case later.

SO(5) with $N_f = 1$. In [19], the matrix model curve $\widehat{y_m}^2$ contains an overall factor x^2. Therefore, there exists no solution for non-degenerated case. We will come to this for degenerated case later.

SO(5) with $N_f = 2$. The SW curve is the same as the curve of SO(4) with $N_f = 1$ which reflects the addition map. Then the factorization problem will lead to the one in SO(4) with $N_f = 1$ theory.

SO(6) with $N_f = 1$. This example is the first nontrivial case. By putting the values $(N, N_1, N_f) = (6,1,1)$ into the curve (2.2), we predict the matrix model curve for this case,

$$y_m^2 = x^2(x^2+3\eta\Lambda^2)^2 - 4\Lambda^6 = (x^2+\eta\Lambda^2)^2(x^2+4\eta\Lambda^2)$$

where η satisfies $\eta^3 = -1$ or equivalently $\eta^6 = 1$. Recall that the matrix model curve for pure SO(5) gauge theory [15] is identical to this curve. That is, $N - N_f = 5$. The factorization problem [19] resulted in the matrix model curve $\widehat{y_m}^2 = x^2(x^2-\alpha^2)^2 - 4\Lambda^6$. There exists a symmetry breaking SO$(6) \to \widehat{\text{SO}(4)} \times$ U(1). For $\alpha^2 = -3\eta\Lambda^2$, this can be factorized as $(x^2+4\eta\Lambda^2)(x^2+\eta\Lambda^2)^2$. For $\alpha^2 = 2\Lambda^2$, the curve $x^2(x^2-\alpha^2)^2 - \alpha^4x^2$ will be $x^4(x^2-4\Lambda^2)$. The latter implies the degenerated case. The characteristic function

[7] One expects that the matrix model curve for U(N) gauge theory with massless flavors (U$(N) \to$ U$(N_1) \times$ U(N_2) where $N = N_1 + N_2$) at $n = 1$ and $n = 2$ singularity takes the form $y_m^2 = [x + \eta\Lambda(2 + c)][x - \eta\Lambda(2-c)](x - \eta\Lambda c)^2$ where $c \equiv \cos(\pi N_2/N - N_f)$ and $\eta^N = 1$.

[8] We denote the curve from strong-coupling approach (factorization problem) by $\widehat{y_m}^2$ and the curve from glueball approach by y_m^2.

$P_6(x) = x^4(x^2+3\eta\Lambda^2)$ becomes $2x^3\rho^3\Lambda^3\mathcal{T}_3(x/2\rho\Lambda)$. These are the vacua surviving when the $\mathcal{N}=2$ theory is perturbed by a quadratic ($n=0$) superpotential and the SO(6) gauge theory becomes massive at low energies.

Let us demonstrate how one can obtain the precise values for (N_0, N_1). The results for a matrix model curve and a characteristic function can be rewritten as follows (we put $\Lambda = 1$ and $\eta = -1$):

$$\widehat{y_m}^2 = \left(x^2-4\right)\left(x^2-1\right)^2, \qquad P_6(x) = x^4\left(x^2-3\right).$$

By using these relations, the function $T(x) = \mathrm{Tr}\frac{1}{x-\Phi}$ is given as [19]

$$T(x) = \frac{P_6'(x) - \frac{(N_f+2)}{x}P_6(x)}{\sqrt{P_6(x)^2-4x^6\Lambda^6}} + \frac{(N_f+2)}{x} = -\frac{3}{\sqrt{x^2-4}} + \frac{3}{x}.$$

There exist three branch cuts on the x-plane $[-2,-1-\epsilon], [-1,1]$, and $[1+\epsilon, 2]$ before taking the limit $x_1 \to x_0 (\epsilon \to 0)$.[9] Since we are assuming $n=0$ and $n=1$ singular case, these three-branch cuts are joined at the locations of $x = \pm 1$ after taking the limit and they reduce to a single-branch cut $[-2,2]$. Therefore, we can explicitly calculate the values (N_0, N_1) as follows:

$$\begin{aligned} N_0 &= \frac{1}{2\pi i}\oint_{A_0} T(x)dx = \frac{2}{2\pi i}\int_{-1}^{1}\left(-\frac{3}{\sqrt{x^2-4}}+\frac{3}{x}\right)dx = \left(\frac{6}{\pi}\int_0^1 \frac{dx}{\sqrt{4-x^2}}\right)+3 = 4, \\ N_1 &= \frac{1}{2\pi i}\oint_{A_1} T(x)dx = \frac{2}{2\pi i}\int_{1}^{2}\left(-\frac{3}{\sqrt{x^2-4}}+\frac{3}{x}\right)dx = \frac{3}{\pi}\int_1^2 \frac{dx}{\sqrt{4-x^2}} = 1 \end{aligned}$$

as we expected.[10]

SO(6) with $N_f = 2$. Once again the factorization problem [19] resulted in the matrix model curve that contains an overall x^2 factor implying a degenerated case.

SO(6) with $N_f = 3$. The factorization problem [19] turned out the matrix model curve that has an overall x^2 factor and is classified as a degenerated case which will be studied later.

SO(7) with $N_f \geq 1$. The curve for SO(7) with N_f flavors is the same as SO(6) curve with (N_f-1) flavors. In particular, since the SW curve of SO(7) theory with $N_f = 1$ is identical to the SO(6) gauge theory with $N_f = 0$, the factorization problem in [14] implies that the matrix model curve is

$$y_m^2 = x^2\left(x^2+4\eta\Lambda^2\right)^2 + 4\eta^2\Lambda^4x^2 + 16\eta^3\Lambda^6 = \left(x^2+2\eta\Lambda^2\right)^2\left(x^2+4\eta\Lambda^2\right), \qquad \eta^4 = 1$$

[9]Recall that a matrix model curve is given by $y_m^2 = \prod_{i=0}^{2}\left(x^2+x_i^2\right)$.

[10]Recently [40], the construction of $T(x)$ on an elliptic curve in terms of three discrete parameters and two continuous ones was studied in the breaking pattern $\mathrm{U}(N) \to \mathrm{U}(N_1)\times\mathrm{U}(N_2)$ where $N = N_1+N_2$. This method will give some hints to obtain this 1-form explicitly without resort to a characteristic function we did here.

where the breaking pattern $SO(7) \to \widehat{SO(5)} \times U(1)$ exists.[11] At the intersections with $n = 0$ branch, the characteristic function $P_6(x)$ can be written in terms of the Chebyshev polynomial of degree 4 with an appropriate identification between these two functions.

SO(8) with $N_f \geq 1$. The curve for SO(8) with $N_f = 1$ can be obtained from the factorization problem. The result is, by applying the method of [19] to the present case here, $\widehat{y_m}^2 = \left(x^2 + \frac{3\pm\sqrt{5}}{2}\eta\Lambda^2\right)^2 \left(x^2 + 4\eta\Lambda^2\right)$ with $\eta^5 = -1$. This curve is exactly the same as the expression from the glueball approach by inserting the values $(N_0, N_1, N_f) = (6, 1, 1)$ into (2.2) where $SO(8) \to \widehat{SO(6)} \times U(1)$. Note that the above matrix model curve is the same as the curve for pure SO(7) gauge theory [15]: $N - N_f = 7$. Although there is no information about the curve with larger flavors from factorization problem, one predicts a matrix model curve through (2.2).

SO(9) with $N_f \geq 1$. The curve for SO(9) with N_f flavors is the same as SO(8) curve with $(N_f - 1)$ flavors. In particular, since the SW curve of SO(9) theory with $N_f = 1$ is identical to the SO(8) gauge theory with $N_f = 0$, the factorization problem in [14] implies that the matrix model curve becomes

$$y_m^2 = x^2 \left(x^2 + 5\eta\Lambda^2\right)^2 + 8\eta^2\Lambda^4 x^2 + 36\eta^3\Lambda^6 = \left(x^2 + 3\eta\Lambda^2\right)^2 \left(x^2 + 4\eta\Lambda^2\right), \qquad \eta^6 = 1$$

where the breaking pattern $SO(9) \to \widehat{SO(7)} \times U(1)$ exists.[12] At the intersections with $n = 0$ branch, the characteristic function $P_8(x)$ can be written in terms of the Chebyshev polynomial of degree 6 with an appropriate identification between these two functions. These are the vacua that survive when the $\mathcal{N} = 2$ theory is perturbed by a quadratic superpotential ($n = 0$).

Recalling the replacement of $(N_0 - 2)$ in pure gauge theory result with $(N_0 - N_f - 2)$ for massless flavors we can get the coupling constant near the singularity without explicit

[11] Let us check how one can determine the precise values for (N_0, N_1). The results for a matrix model curve and a characteristic function are given by (we put $\Lambda = 1$ and $\eta = -1$): $\widehat{y_m}^2 = (x^2-4)(x^2-2)^2, P_6(x) = x^2(x^4-4x^2+2)$. By using these relations, the function $T(x)$ is given by $T(x) = \frac{P_6'(x) - \frac{(N_f+1)}{x}P_6(x)}{\sqrt{P_6(x)^2 - 4x^4\Lambda^8}} + \frac{(N_f+2)}{x} = -\frac{4}{\sqrt{x^2-4}} + \frac{3}{x}$. Note that there was a typo in $T(x)$ [14] and there should be present an extra $1/x$ term in $T(x)$. There exist three-branch cuts on the x-plane $[-2, -\sqrt{2}-\epsilon], [-\sqrt{2}, \sqrt{2}]$, and $[\sqrt{2}+\epsilon, 2]$ before taking the limit $x_1 \to x_0 (\epsilon \to 0)$. These three-branch cuts are joined at the locations of $x = \pm\sqrt{2}$ after taking the limit and they reduce to a single-branch cut $[-2, 2]$. Therefore, we can explicitly compute the values (N_0, N_1) as follows: $N_0 = \frac{2}{2\pi i}\int_{-\sqrt{2}}^{\sqrt{2}} \left(-\frac{4}{\sqrt{x^2-4}} + \frac{3}{x}\right) dx = 5, N_1 = \frac{2}{2\pi i}\int_{\sqrt{2}}^{2} \left(-\frac{4}{\sqrt{x^2-4}} + \frac{3}{x}\right) dx = 1$.

[12] By using $\widehat{y_m}^2 = \left(x^2-4\right)\left(x^2-3\right)^2, P_8(x) = x^4\left(x^2-3\right)^2 - 2x^2$, the function $T(x)$ is given by $T(x) = \frac{P_8'(x) - \frac{(N_f+1)}{x}P_8(x)}{\sqrt{P_8(x)^2 - 4x^4\Lambda^8}} + \frac{(N_f+2)}{x} = -\frac{6}{\sqrt{x^2-4}} + \frac{3}{x}$. There exist three-branch cuts on the x-plane $[-2, -\sqrt{3} - \epsilon], [-\sqrt{3}, \sqrt{3}]$, and $[\sqrt{3}+\epsilon, 2]$ before taking the limit $x_1 \to x_0 (\epsilon \to 0)$. These three-branch cuts are joined at the locations of $x = \pm\sqrt{3}$ after taking the limit and they reduce to a single-branch cut $[-2, 2]$. Therefore, we can explicitly compute the values (N_0, N_1) as follows: $N_0 = \frac{2}{2\pi i}\int_{-\sqrt{3}}^{\sqrt{3}} \left(-\frac{6}{\sqrt{x^2-4}} + \frac{3}{x}\right) dx = 7, N_1 = \frac{2}{2\pi i}\int_{\sqrt{3}}^{2} \left(-\frac{6}{\sqrt{x^2-4}} + \frac{3}{x}\right) dx = 1$.

JHEP02(2004)009

calculation,

$$\frac{1}{2\pi i}\tau_{ij} = \frac{\partial^2 \mathcal{F}_p(S_k)}{\partial S_i \partial S_j} - \delta_{ij}\frac{1}{N_i}\sum_{l=1}^{n} N_l \frac{\partial^2 \mathcal{F}_p(S_k)}{\partial S_i \partial S_l} - \delta_{ij}\left(\frac{N_0 - N_f - 2}{N_i}\right)\frac{\partial^2 \mathcal{F}_p(S_k)}{\partial S_0 \partial S_i},$$

where $i, j = 1, 2, \cdots, n$. In the quartic tree-level superpotential case ($n = 1$) we are considering, there is only one coupling constant and it is given by, due to the cancellation of first two terms above,

$$\frac{1}{2\pi i}\tau = -\frac{i\pi}{16}\frac{(N_0 - N_f - 2)^2}{(N - N_f - 2)^2}\frac{1}{\log\left(\frac{16}{1-k'^2}\right)},$$

where $k'^2 = x_0^2(x_2^2 - x_1^2)/x_1^2(x_2^2 - x_0^2)$. Since we are assuming asymptotically free gauge theory, the quantity $(N - N_f - 2)$ is greater than zero. When the condition $(N_0 - N_f - 2) = 0$ is satisfied, τ becomes zero. In this case, the asymptotic freedom for $\mathrm{SO}(N_0)$ gauge theory breaks down. Therefore the situation is the same as the one-cut case (equivalently $n = 0$ case) in which the coupling constant is trivially zero. Our result gives a consistency.

3. The curve for degenerated SO(N) gauge theory with massless flavors

In the non-degenerated case, every root of $W_3'(x)$ possess at least one D5 brane wrapping around it. For the degenerated case, some roots do not contain wrapping D5 branes around them. At the classical limits, there are symmetry breakings characterized by

$$\mathrm{SO}(N) \to \widehat{\mathrm{SO}(N_0)} \times \mathrm{U}(N_1)\,, \qquad \mathrm{SO}(N) \to \mathrm{U}\left(\left[\frac{N}{2}\right]\right), \qquad \mathrm{SO}(N) \to \widehat{\mathrm{SO}(N)}\,.$$

The last pattern occurs only if further constraints are imposed on the first two breaking patterns. For degenerated case we can write the matrix model curve which is valid on the whole degenerated branch. As we did for pure gauge theory, the dual periods are defined by the integrals over x and the equation of motion of a single field F provides one relation. Recalling the simple replacements $(N - 2) \to (N - N_f - 2)$ and $(N_0 - 2) \to (N_0 - N_f - 2)$ we made before, we can obtain the matrix model curve[13] as

$$y_{m,d}^2 = \left(x^2 + m\right)^2 + 4F\,, \qquad m = \frac{K^2 - F}{K}\,, \qquad K \equiv \left[\frac{(-\Lambda^2)^{N-N_f-2}}{(-F)^{N_1}}\right]^{\frac{1}{N_0 - N_f - 2}}. \quad (3.1)$$

Here we rescaled the function K and corrected a typo appeared in [15]. This formula depends on the parameter F, N_0, N_1, and N_f. Turning on a field F to the particular values

[13] For the degenerated case also, the relation between $W_3'(x)$ and $F_4(x)$ can be written as

$$y_{m,d}^2 = F_4(x) = \left(\frac{W_3'(x)}{x}\right)^2 + 4F\,, \qquad N_f < N - 3\,,$$
$$y_{m,d}^2 = F_4(x) = \left(\frac{W_3'(x)}{x}\right)^2 + 4F - 4x^2\Lambda^2\,, \qquad N_f = N - 3\,.$$

In particular, when $N_f = N - 3$, the matrix model curve $y_{m,d}^2$ has an extra contribution from $4x^2\Lambda^2$. Therefore in this particular case, the curve should be $y_{m,d}^2 = \left(x^2 + m - 2\Lambda^2\right)^2 + 4F$ and $m - 2\Lambda^2 = \frac{K^2-F}{K}$.

will reduce to the last breaking pattern $SO(N) \to \widehat{SO(N)}$ above. When $(N_0 - N_f - 2) = 0$ (we will meet this situation in the examples below), the above formula becomes ill-defined. We go back to the derivation of the formula[14] and can find another formula for this special case,

$$y_{m,d}^2 = \left(x^2 + D\right)^2 - \frac{4\Lambda^4}{\eta^2}, \qquad \eta^{2N_1} = 1, \qquad \text{for } (N_0 - N_f - 2) = 0 \tag{3.2}$$

where D is arbitrary parameter.

To demonstrate these general solutions (3.1) and (3.2), let us consider $SO(N)$ gauge theories with $N_f (\leq N-3)$ flavors where $N = 4, 5$ or 6.

$SO(4)$ **with** $N_f = 1$. There are two breaking patterns on the degenerated case: $\widehat{SO(2)} \times U(1)$ and $U(2)$.

1. For the breaking pattern $SO(4) \to \widehat{SO(2)} \times U(1)$, by substituting the values $(N, N_1, N_f) = (4, 1, 1)$ into (3.1) with the footnote 13 (recall that $N = N_0 + 2N_1$), the matrix model curve from glueball approach can be represented by

$$y_{m,d}^2 = \left(x^2 + m - 2\Lambda^2\right)^2 + 4F, \qquad m = \frac{F}{\Lambda^2} + \Lambda^2.$$

We find that the solutions given in [19] perfectly agree with the above result for $y_{m,d}^2$ from glueball approach. In other words, the factorization problem characterized by $P_4^2(x) - 4x^6\Lambda^2 = H_2^2(x)[(x^2 - D)^2 + 4F]$ reproduces the above solution where $D = \Lambda^2 - \frac{F}{\Lambda^2}$ (For the notation of [19], a is equal to $-2D$ and $(\alpha^2 + 2\Lambda^2)$ corresponds to $\Lambda^2 - \frac{F}{\Lambda^2} = D$). Then $\alpha^2 = -\Lambda^2 - \frac{F}{\Lambda^2}$ which is equal to $-m$. For $F = 0$, there exists a Special branch with $SO(4) \to \widehat{SO(4)}$ and for $(m - 2\Lambda^2)^2 + 4F = 0$, there exists a Chebyshev branch with $SO(4) \to \widehat{SO(4)}$.

2. For the breaking pattern $SO(4) \to U(2)$, we obtain the following matrix model curve from the glueball approach, by plugging the values $(N, N_1, N_f) = (4, 2, 1)$ into (3.1) with the footnote 13,

$$y_{m,d}^2 = (x^2 + m - 2\Lambda^2)^2 + 4F, \qquad m = \left(-\frac{F^2}{\Lambda^2}\right)^{\frac{1}{3}} - F\left(-\frac{\Lambda^2}{F^2}\right)^{\frac{1}{3}} + 2\Lambda^2.$$

Again by using the different parametrization (as in previous case) from the one in [19], we can obtain the same matrix model curve with D which is equal to $-m + 2\Lambda^2$ above where the relation $6\alpha^2\Lambda^2 + 14\Lambda^4 - 2(\alpha^2 + 3\Lambda^2)\sqrt{4\alpha^2\Lambda^2 + 9\Lambda^4} = 4F$ (corresponding to $b - a^2/4$ in the notation of [19]) will provide one solution for $\alpha^2 = -m$ in the above. Therefore, the results obtained from two approaches are equivalent to each other.

[14]In this case, there is a relation $2N_1 \log \left|\frac{4\Lambda^2}{b^2 - a^2}\right| = 0$. By putting $\eta = \pm\frac{4\Lambda^2}{b^2 - a^2}$ and realizing $D = \frac{1}{2}(a^2 + b^2)$, one gets the above formula.

JHEP02(2004)009

SO(5) with $N_f = 1$. There are two breaking patterns on the degenerated case: $\widehat{\mathrm{SO}(3)} \times \mathrm{U}(1)$ and $\widehat{\mathrm{SO}(1)} \times \mathrm{U}(2)$.

1. For the former case $\mathrm{SO}(5) \to \widehat{\mathrm{SO}(3)} \times \mathrm{U}(1)$, since the condition $(N_0 - N_f - 2) = 0$ is satisfied, the matrix model curve can be read off, from (3.2),

$$y_{m,d}^2 = \left(x^2 + D\right)^2 - 4\Lambda^4 \,.$$

This is exactly the first kind of solution given in [19] and the D here corresponds to $a/2 = -\alpha^2$ there and $-4\Lambda^4$ corresponds to $(b - a^2/4)$. For $D^2 = 4\Lambda^4$, there is a Chebyshev vacuum where $\mathrm{SO}(5) \to \widehat{\mathrm{SO}(5)}$.

2. For the second breaking pattern $\mathrm{SO}(5) \to \widehat{\mathrm{SO}(1)} \times \mathrm{U}(2)$, by paying attention to the value of N_0, we simply put $N_0 = 1$ and find the following matrix model curve,

$$y_{m,d}^2 = \left(x^2 + m\right)^2 + 4F \,, \qquad m = \frac{\eta F}{\Lambda^2} - \eta\Lambda^2$$

where η is 2-nd root of unity. These solutions are exactly the same as the ones from factorization problem in [19] ($m = a/2$ and $b - a^2/4 = 4F$). For $a = \pm 2\Lambda^2$ (+ sign corresponds to $\eta = -1$ above and $-$ sign corresponds to $\eta = 1$), there is a Special vacuum where $\mathrm{SO}(5) \to \widehat{\mathrm{SO}(5)}$. This curve is the same as the matrix model curve for pure degenerated SO(4) gauge theory [15] ($N - N_f = 4$) where $\mathrm{SO}(4) \to \mathrm{U}(2)$.

SO(6) with $N_f = 1$. In this case, there are three kinds of breaking patterns on the degenerated case: $\widehat{\mathrm{SO}(4)} \times \mathrm{U}(1)$, $\widehat{\mathrm{SO}(2)} \times \mathrm{U}(2)$, and U(3).

1. For the breaking pattern $\mathrm{SO}(6) \to \widehat{\mathrm{SO}(4)} \times \mathrm{U}(1)$, the matrix model curve is represented as from (3.1) with $(N, N_1, N_f) = (6, 1, 1)$

$$y_{m,d}^2 = \left(x^2 + m\right)^2 + 4F, \qquad m = \frac{\Lambda^6}{F} - \frac{F^2}{\Lambda^6} \,.$$

Identifying $4F$ here with b in [19] (and $m = -a$), we can see the exact agreement. Recall that this curve is identical to the matrix model curve for pure degenerated SO(5) gauge theory [15] ($N - N_f = 5$) where $\mathrm{SO}(5) \to \mathrm{SO}(3) \times \mathrm{U}(1)$.

2. On the other hand, for the breaking pattern $\mathrm{SO}(6) \to \mathrm{U}(3)$, the matrix model curve can be written as from (3.1) together with $(N, N_1, N_f) = (6, 3, 1)$

$$y_{m,d}^2 = \left(x^2 + m\right)^2 + 4F, \qquad m = \frac{\eta F}{\Lambda^2} - \eta^2\Lambda^2$$

where $\eta^3 = 1$. By identifying (a, b) with $(-m, 4F)$, we can see the complete agreement with the first, second and third solutions of the equation (6.38) in [19] although we have not written them explicitly. For $b = 4F = 0$, this leads to a Special vacuum of $\mathrm{SO}(6) \to \widehat{\mathrm{SO}(6)}$.

3. For the breaking pattern $\mathrm{SO}(6) \to \widehat{\mathrm{SO}(2)} \times \mathrm{U}(2)$, the solution from factorization was not new one but can be written as the previous solution we have discussed already.

$\mathrm{SO}(6)$ **with** $N_f = 2$.

1. For the breaking pattern $\mathrm{SO}(6) \to \widehat{\mathrm{SO}(4)} \times \mathrm{U}(1)$, the matrix model curve can be written as from (3.2)
$$y_{m,d}^2 = \left(x^2 + D\right)^2 - 4\Lambda^4 .$$
This corresponds to the first kind solution in [19]. When $D^2 = 4\Lambda^4$, this will lead to a Chebyshev vacuum with $\mathrm{SO}(6) \to \widehat{\mathrm{SO}(6)}$.

2. For the breaking pattern $\mathrm{SO}(6) \to \widehat{\mathrm{SO}(2)} \times \mathrm{U}(2)$, the matrix model curve is given by from (3.1) with $(N, N_1, N_f) = (6, 2, 2)$
$$y_{m,d}^2 = \left(x^2 + m\right)^2 + 4F , \qquad m = \frac{\eta F}{\Lambda^2} - \eta\Lambda^2$$
where η is 2-nd root of unity. This can be seen from the degenerated pure $\mathrm{SO}(4)$ gauge theory with the breaking pattern $\mathrm{SO}(4) \to \mathrm{U}(2)$ [15] through the addition map. If we change the parametrization by $4F = 4\eta a\Lambda^2 + 4\Lambda^4$, the above matrix model curve can be rewritten as
$$y_{m,d}^2 = \left(x^2 + a\right)^2 + 4\eta\Lambda^2 x^2 ,$$
which is exactly the same as the second kind of solution in [19]. For $a = 0$, it gives a Chebyshev vacuum of $\mathrm{SO}(6) \to \widehat{\mathrm{SO}(6)}$. For $a = -\eta\Lambda^2$, a Special branch appears.[15]

$\mathrm{SO}(6)$ **with** $N_f = 3$. In this case, there are three breaking patterns: $\widehat{\mathrm{SO}(4)} \times \mathrm{U}(1)$, $\widehat{\mathrm{SO}(2)} \times \mathrm{U}(2)$ and $\mathrm{U}(3)$.

1. For the first two cases, we expect that the matrix model curves are the same as the ones in $\mathrm{SO}(4)$ theory with $N_f = 1$ as discussed above, which comes from the addition map for massless case. For the breaking pattern $\mathrm{SO}(6) \to \widehat{\mathrm{SO}(4)} \times \mathrm{U}(1)$, the matrix model curve can be represented as from (3.1) with the footnote 13
$$y_{m,d}^2 = \left(x^2 + m - 2\Lambda^2\right)^2 + 4F, \qquad m = \frac{F}{\Lambda^2} + \Lambda^2 .$$
From the result of [19], the matrix model curve was $\widehat{y_m}^2 = (x^2 - \alpha^2 - 2\Lambda^2)^2 - 4\alpha^2 - 4\Lambda^4$. By introducing a new field $4F = -4\alpha^2\Lambda^2 - 4\Lambda^4$, one can identify $\alpha^2 = -\Lambda^2 - \frac{F}{\Lambda^2}$ which is equal to $-m$ in the above.

2. For the breaking pattern, $\mathrm{SO}(6) \to \widehat{\mathrm{SO}(2)} \times \mathrm{U}(2)$, from (3.1) with the footnote 13 the matrix model curve can be obtained, with $(N, N_1, N_f) = (6, 2, 3)$,
$$y_{m,d}^2 = \left(x^2 + m - 2\Lambda^2\right)^2 + 4F, \qquad m = \left(-\frac{F^2}{\Lambda^2}\right)^{\frac{1}{3}} - F\left(-\frac{\Lambda^2}{F^2}\right)^{\frac{1}{3}} + 2\Lambda^2 .$$

[15] For the breaking pattern $\mathrm{SO}(6) \to \mathrm{U}(3)$, we can predict the following matrix model curve, from (3.1) with $(N, N_1, N_f) = (6, 3, 2)$, $y_{m,d}^2 = \left(x^2 + m\right)^2 + 4F$, and $m = \frac{\eta F^{\frac{3}{4}}}{\Lambda} - \frac{1}{\eta}\Lambda F^{\frac{1}{4}}$ where $\eta^8 = 1$. In this case, the factorization problem becomes very complicated. Since we cannot obtain the explicit results for the matrix model curve from the factorization, we have not checked them in this example.

As expected, this matrix model curve is the same as the one in SO(4) theory with $N_f = 1$ case and agrees with the one from the factorization problem. In other words, the factorization problem $P_6^2(x) - 4x^4\Lambda^4 = x^4 H_2^2(x)[(x^2 - D)^2 + 4F]$ reproduces the above solution where $D = -(-F^2/\Lambda^2)^{1/3} + F(-\Lambda^2/F^2)^{1/3}$. When $F = 0$, it gives a Special vacuum of $\mathrm{SO}(6) \to \widehat{\mathrm{SO}(6)}$. For $(m - 2\Lambda^2)^2 + 4F = 0$, it leads to a Chebyshev vacuum of $\mathrm{SO}(6) \to \widehat{\mathrm{SO}(6)}$.[16]

In summary, for the SO(N) gauge theories with $N_f(\leq N-3)$ flavors where $N = 4, 5$ or 6, we have convinced ourselves that the solutions (3.1) and (3.2) coincide with the matrix model curve from the strong-coupling approach except that we could not check the SO(6) gauge theory with $N_f = 2, 3$ where there exists a breaking pattern $\mathrm{SO}(6) \to \mathrm{U}(3)$ (in this case, $N_0 = 0$). The main obstacle why the factorization problem for $\mathrm{SO}(N) \to \mathrm{U}([N/2])$ becomes so complicated was the SW curve does not possess any power of x^2, contrary to the factorization problem for $\mathrm{SO}(N) \to \widehat{\mathrm{SO}(N_0)} \times \mathrm{U}(N_1), N_0 \neq 0$ in which the SW curve has a power of x^2 (either Chebyshev or Special branch) making the factorization easier. Therefore, at least one expects that our matrix model curve (3.1) and (3.2) will predict the solutions for the degenerated SO(N) gauge theories with $N_f(\leq N-3)$ massless flavors where $N \geq 7$ and $\mathrm{SO}(N) \to \widehat{\mathrm{SO}(N_0)} \times \mathrm{U}(N_1), N_0 \neq 0$.

4. The $n = 0$ and $n = 1$ singularity: USp($2N$) gauge theory with massless flavors

As in the SO(N) case discussed previous section, we can obtain the matrix model curve without explicit calculation by looking at the pure USp($2N$) gauge theory. We only have to replace the expression $(2N + 2)$ in the result of [15] for the matrix model curve with $(2N - N_f + 2)$. With this replacement in mind, we immediately obtain the matrix model curve near the singularity on the non-degenerated case from the glueball approach:[17]

$$y_m^2 = x^2\left(x^2 + m\right)^2 + \langle f_2 \rangle x^2 + \langle f_0 \rangle = \left[x^2 + 2\eta\Lambda^2\left(1 + c\right)\right]^2\left(x^2 + 4\eta\Lambda^2\right),$$

where the fields, parameter of a superpotential, and glueball field are given by

$$\begin{aligned} \langle f_2 \rangle &= 4\eta^2\Lambda^4\left(1 + 2c\right), & \langle f_0 \rangle &= 16\eta^3\Lambda^6\left(1 + c\right)^2, \\ m &= 2\eta\Lambda^2\left(2 + c\right), & \langle S \rangle &= -\eta^2\Lambda^4\left(1 + 2c\right), \qquad c \equiv \cos\left(\frac{2\pi N_1}{2N - N_f + 2}\right). \end{aligned} \tag{4.1}$$

[16] The last one is the $\mathrm{SO}(6) \to \mathrm{U}(3)$ breaking pattern. In this case the matrix model curve is given by, from (3.1) and the footnote 13 with $(N, N_1, N_f) = (6, 3, 3)$, $y_{m,d}^2 = \left(x^2 + m - 2\Lambda^2\right)^2 + 4F$, and $m = \left(\frac{F^3}{\Lambda^2}\right)^{\frac{1}{5}} - F\left(\frac{\Lambda^2}{F^3}\right)^{\frac{1}{5}} + 2\Lambda^2$. Since the factorization problem becomes very complicated, we have not checked them in this example.

[17] The effective glueball superpotential for massless flavors is $W_{\rm eff} = 2\pi i\left[(2N_0 + 2 - N_f)\Pi_0 + 2N_1\Pi_1\right] - 2(2N + 2 - N_f)S\log\left(\frac{\Lambda}{\Lambda_0}\right)$.

Here η is the $(2N - N_f + 2)$-th root of unity, that is $\eta^{2N-N_f+2} = 1$.[18] This matrix model curve with massless flavors is exactly the same as pure case with the replacement $(2N+2) \to (2N - N_f + 2)$. Note that the above solutions have the special combination $(2N - N_f + 2)$ which reflects the addition map discussed in [20], relating the Chebyshev branches (or the Special branches) of two different gauge groups. Namely, the curve of USp$(2N)$ with N_f massless flavors can be reduced to the curve of USp$(2N-2r)$ with $(N_f - 2r)$ massless flavors.

Let us demonstrate these general features (4.1) explicitly by comparing them with the results from strong-coupling approach obtained in [20] already. Let us consider USp$(2N)$ gauge theories with $N_f(\leq 2N+1)$ flavors where $N = 2$ or 3.

USp(4) **with** $N_f = 1$. From the results in [20], the matrix model curve has an extra double root at $d = \frac{3\pm\sqrt{5}}{2}\Lambda^2\epsilon$ on the non-degenerated case. It was factorized as $\widehat{y_m}^2 = (t + \frac{3\pm\sqrt{5}}{2}\epsilon\Lambda^2)^2(t+4\epsilon\Lambda^2)$ where ϵ is 5-th root of unity.[19] On the other hand, the matrix model curve from the glueball approach becomes, from (4.1) with $(N, N_1, N_f) = (2,1,1)$,

$$y_m^2 = \left(x^2 + \frac{3\pm\sqrt{5}}{2}\eta\Lambda^2\right)^2 (x^2 + 4\eta\Lambda^2)$$

where the other values in different parametrization are given by

$$m = \frac{(7\pm\sqrt{5})}{2}\eta\Lambda^2\,, \qquad \langle f_2\rangle = 2(1\pm\sqrt{5})\eta^2\Lambda^4\,, \qquad \langle f_0\rangle = 2(7\pm 3\sqrt{5})\eta^3\Lambda^6\,.$$

Here η is 5-th root of unity. Now it is easy to see the exact agreement between two results.

USp(4) **with** $N_f = 2$. The matrix model curve obtained from the factorization problem can be represented, at singular point, as $\widehat{y_m}^2 = (t+2\eta\Lambda^2)^2(t+4\eta\Lambda^2)$ where η is 4-th root of unity (Note that there is a typo in [20]). The characteristic function $B_6(x) = x^2(x^4 + 4\eta\Lambda^2x^2 + 2\eta^2\Lambda^4)$ will become $2\rho^2x^2\Lambda^4\mathcal{T}_2(\frac{x^2}{2\rho\Lambda^2}+1)$ with $\rho^4 = 1$ by identifying $\rho = \eta$. These are the vacua surviving when the $\mathcal{N}=2$ theory is perturbed by a quadratic $(n=0)$ superpotential and the USp(4) gauge theory becomes massive at low energies. To see the correctness of our formula (4.1) we put simply the values $(2N_0, N_1, N_f) = (2,1,2)$ (Note that $2N = 2N_0 + 2N_1$) where USp$(4) \to \widehat{\text{USp}(2)} \times$ U(1) into the formula (4.1) and get the following matrix model curve,

$$y_m^2 = x^2(x^2+4\eta\Lambda^2)^2 + 4\eta^2\Lambda^4x^2 + 16\eta^3\Lambda^6 = (x^2+2\eta\Lambda^2)^2(x^2+4\eta\Lambda^2)\,.$$

Now there exists a perfect agreement.

[18] By recognizing the matrix model curve for USp$(2N)$ gauge theory characterized by $y_m^2 = \prod_{i=0}^{2}(x^2 - x_i^2)$, one can read off the matrix model curve for USp$(2N)$ with flavors from the curve for SO(N) gauge theory with flavors by replacing $(N-2) \to (2N+2), x_i^2 \to -x_i^2$ where $i = 0,1,2$ as follows: $x_2^2 = -4\eta\Lambda^2$, and $x_0^2 = -2\eta\Lambda^2(1+c)$ where η satisfies the condition $\eta^{2N-N_f+2} = 1$ for $(2N - N_f + 2)$ even and $\eta^{2N-N_f+2} = -1$ for $(2N - N_f + 2)$ odd. When we define $\epsilon \equiv -\eta$ it satisfies $\epsilon^{2N-N_f+2} = 1$ for both cases. The solutions can be rewritten as $x_2^2 = 4\epsilon\Lambda^2$, $x_0^2 = 2\epsilon\Lambda^2(1+c)$, and $\epsilon^{2N-N_f+2} = 1$ where c is given in (4.1). This is exactly the same as (4.1) by using the different parametrization.

[19] We use a relation $t = x^2$ all the time in this paper.

Let us demonstrate how one can determine the precise values for $(2N_0, N_1)$. The results can be rewritten as follows (we put $\Lambda = 1$ and $\eta = 1$):

$$\widehat{y_m}^2 = (x^2+4)(x^2+2)^2 \,, \qquad B_6(x) = x^2(x^4+4x^2+2) \,.$$

By using these relations, the function $T(x) = \mathrm{Tr}\frac{1}{x-\Phi}$ is given as [20]

$$T(x) = \frac{B_6'(x) - \frac{N_f}{x}B_6(x)}{\sqrt{B_6(x)^2 - 4x^4\Lambda^8}} + \frac{(N_f-2)}{x} = \frac{4}{\sqrt{x^2+4}} \,.$$

There exist three branch cuts on the x-plane $[-2i, -(\sqrt{2}+\epsilon)i], [-\sqrt{2}i, \sqrt{2}i]$, and $[(\sqrt{2}+\epsilon)i, 2i]$ before taking the limit $x_1 \to x_0$. Since we are assuming $n=0$ and $n=1$ singular case, these three-branch cuts are joined at the locations of $x = \pm\sqrt{2}i$ after taking the limit and they reduce to a single-branch cut $[-2i, 2i]$. Therefore, we can explicitly calculate the values $(2N_0, N_1)$ as follows:

$$\begin{aligned} 2N_0 &= \frac{1}{2\pi i}\oint_{A_0} T(x)dx = \frac{2}{2\pi i}\int_{-\sqrt{2}i}^{\sqrt{2}i} \frac{4}{\sqrt{x^2+4}}dx = 2 \,, \\ N_1 &= \frac{1}{2\pi i}\oint_{A_1} T(x)dx = \frac{2}{2\pi i}\int_{\sqrt{2}i}^{2i} \frac{4}{\sqrt{x^2+4}}dx = 1 \end{aligned}$$

as we expected.

$\mathrm{USp}(4)$ **with** $N_f = 3$. The matrix model curve at the singularity was given in [20] as $\widehat{y_m}^2 = (t+\eta\Lambda^2)^2(t+4\eta\Lambda^2)$ where $\eta^3 = 1$. Note that α^2 corresponds to $-3\eta\Lambda^2$. The $B_6(x)$ is equal to $2\rho^3x^3\Lambda^3\mathcal{T}_3(x/2\rho\Lambda)$ by putting $\rho = \eta$. At these intersection points, the vacua survive when the $\mathcal{N}=2$ theory is perturbed by a quadratic $(n=0)$ superpotential. If we put the values $(2N_0, N_1, N_f) = (2,1,3)$ where $\mathrm{USp}(4) \to \widehat{\mathrm{USp}(2)} \times \mathrm{U}(1)$ into our formula (4.1) we obtain the matrix model curve at the singularity,

$$y_m^2 = x^2(x^2+3\eta\Lambda^2)^2 + 4\eta^3\Lambda^6 = (x^2+\eta\Lambda^2)^2(x^2+4\eta\Lambda^2) \,.$$

We see the agreement between two approaches.

$\mathrm{USp}(4)$ **with** $N_f = 4$. There is no non-degenerated case and we will discuss it in the degenerated case later.

$\mathrm{USp}(4)$ **with** $N_f = 5$. In [20], the factorization problem resulted in the matrix model curve $\widehat{y_m}^2$ which contains an overall factor x^2. Therefore, there exists no solution for non-degenerated case. We will come to this case for degenerated case later.

$\mathrm{USp}(6)$ **with** $N_f = 2$.

1. For the first solution $\mathrm{USp}(6) \to \widehat{\mathrm{USp}(4)} \times \mathrm{U}(1)$ in [20], the matrix model curve has an extra double root at $d = -\epsilon\Lambda^2$, where ϵ is 6-th root of unity. At the singular points, the matrix model can be factorized as $\widehat{y_m}^2 = \left(t - 3\epsilon\Lambda^2\right)^2\left(t - 4\epsilon\Lambda^2\right)$. If we simply

put the values $(2N_0, N_1, N_f) = (4, 1, 2)$ into our general formula (4.1) for the matrix model curve at the singularities, we obtain the following results,

$$y_m^2 = x^2(x^2 + 5\eta\Lambda^2)^2 + 8\eta^2\Lambda^4 x^2 + 36\eta^3\Lambda^6 = (x^2 + 3\eta\Lambda^2)^2(x^2 + 4\eta\Lambda^2)$$

where η is 6-th root of unity. If we identify this η with $-\epsilon$ in the solution for factorization problem, we see the agreement of two approaches.[20] This choice for $B_8(x)$ can be written as a Chebyshev polynomial of degree 3 with appropriate identification. At these intersection points, the vacua survive when the $\mathcal{N} = 2$ theory is perturbed by a quadratic $(n = 0)$ superpotential. By the addition map, the above matrix model curve from glueball approach can be obtained also from the pure USp(4) gauge theory $(2N - N_f = 4)$ with the breaking pattern USp(4) $\rightarrow$ USp(2) $\times$ U(1) [15].

2. For the second solution USp(6) $\rightarrow \widehat{\mathrm{USp}(2)} \times$ U(2), the matrix model curve $\widehat{y_m}^2 = t(t+d)^2 - 4\eta\Lambda^6$ has an extra double root at $d = -3\epsilon\Lambda^2$, $\epsilon^6 = 1$ and it can be factorized as $\widehat{y_m}^2 = (t - \epsilon\Lambda^2)^2(t - 4\epsilon\Lambda^2)$ with $\eta = \epsilon^3$. If we simply put the values $(2N_0, N_1, N_f) = (2, 2, 2)$ into our general formula (4.1) for the matrix model curve at the singularities, we obtain the following results,

$$y_m^2 = x^2(x^2 + 3\eta\Lambda^2)^2 + 4\eta^3\Lambda^6 = (x^2 + \eta\Lambda^2)^2(x^2 + 4\eta\Lambda^2)\,,$$

where η is 6-th root of unity. If we identify this η with $-\epsilon$ in the solution for factorization problem, we see the agreement of two approaches.[21] At the intersection points, the vacua survive when the $\mathcal{N} = 2$ theory is perturbed by a quadratic $(n = 0)$ superpotential.

USp(6) **with** $N_f = 4$**.** The factorization problem [20] turned out that the matrix model curve has an overall x^2 factor and is classified as a degenerated case which will be studied later.

USp(6) **with** $N_f = 6$**.** Again the factorization problem [20] turned out the matrix model curve has an overall x^2 factor and it is classified as a degenerated case which will be studied later.

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[20] By using $\widehat{y_m}^2 = \left(x^2+4\right)\left(x^2+3\right)^2$, and $B_8(x) = x^2\left(x^6+6x^4+9x^2+2\right)$, the function $T(x)$ is given by $T(x) = \frac{B_8'(x) - \frac{N_f}{x}B_8(x)}{\sqrt{B_8(x)^2 - 4x^4\Lambda^8}} + \frac{(N_f-2)}{x} = \frac{6}{\sqrt{x^2+4}}$. There exist three branch cuts on the x-plane $[-2i, -(\sqrt{3}+\epsilon)i]$, $[-\sqrt{3}i, \sqrt{3}i]$, and $[(\sqrt{3}+\epsilon)i, 2i]$ before taking the limit $x_1 \rightarrow x_0$. These three-branch cuts are joined at the locations of $x = \pm\sqrt{3}i$ after taking the limit and they reduce to a single-branch cut $[-2i, 2i]$. Therefore, we can explicitly calculate the values $(2N_0, N_1)$ as follows: $2N_0 = \frac{1}{2\pi i}\oint_{A_0} T(x)dx = \frac{2}{2\pi i}\int_{-\sqrt{3}i}^{\sqrt{3}i} \frac{6}{\sqrt{x^2+4}}dx = 4$, and $N_1 = \frac{1}{2\pi i}\oint_{A_1} T(x)dx = \frac{2}{2\pi i}\int_{\sqrt{3}i}^{2i} \frac{6}{\sqrt{x^2+4}}dx = 1$ as we expected.

[21] By using $\widehat{y_m}^2 = \left(x^2+4\right)\left(x^2+1\right)^2$, and $B_8(x) = x^2\left(x^6+6x^4+9x^2+2\right)$, the function $T(x)$ is given by $T(x) = \frac{6}{\sqrt{x^2+4}}$. Therefore, we can explicitly calculate the values $(2N_0, N_1)$ as follows: $2N_0 = \frac{2}{2\pi i}\int_{-i}^{i} \frac{6}{\sqrt{x^2+4}}dx = 2$, and $N_1 = \frac{2}{2\pi i}\int_{i}^{2i} \frac{6}{\sqrt{x^2+4}}dx = 2$.

JHEP02(2004)009

As in the SO(N) case we can obtain the coupling constant by using simple replacement of $(2N_0+2)$ in pure gauge theory result with $(2N_0-N_f+2)$ we can get the coupling constant near the singularity without explicit calculation,

$$\frac{1}{2\pi i}\tau_{ij} = \frac{\partial^2 \mathcal{F}_p(S_k)}{\partial S_i \partial S_j} - \delta_{ij}\frac{1}{N_i}\sum_{l=1}^{n} N_l \frac{\partial^2 \mathcal{F}_p(S_k)}{\partial S_i \partial S_l} - \delta_{ij}\left(\frac{2N_0-N_f+2}{N_i}\right)\frac{\partial^2 \mathcal{F}_p(S_k)}{\partial S_0 \partial S_i},$$

where $i,j=1,2,\cdots,n$. In the quartic tree-level superpotential case ($n=1$), there is only one coupling constant and it is given by,

$$\frac{1}{2\pi i}\tau = -\frac{i\pi}{16}\frac{(2N_0-N_f+2)^2}{(2N-N_f+2)^2}\frac{1}{\log\left(\frac{16}{1-k'^2}\right)},$$

where $k'^2 = x_0^2(x_2^2-x_1^2)/x_1^2(x_2^2-x_0^2)$. When the condition $(2N_0-N_f+2)=0$ is satisfied, τ becomes zero. As discussed SO(N) case, the asymptotic freedom for USp($2N_0$) gauge theory breaks down and effectively the situation is the same as $n=0$ case. Our result gives a consistency with $n=0$ case.

5. The curve for degenerated USp($2N$) gauge theory with massless flavors

Again, we want to get a matrix model curve for a degenerated USp($2N$) case from the degenerated SO(N) results discussed in previous section. From the matrix model curve for SO(N) gauge theory, $y_{m,d}^2=(x^2+a^2)(x^2+b^2)$, by simply replacing (a^2,b^2) of SO(N) with $(-a^2,-b^2)$, we obtain the matrix model curve for USp($2N$) gauge theory. In other words, we replace the parameter m of a superpotential in SO(N) with $-m$. Therefore, our matrix model curve[22] becomes

$$y_{m,d}^2 = (x^2-m)^2+4F\,, \qquad m=\frac{K^2-F}{K}\,, \qquad K\equiv\left[\frac{(-\Lambda^2)^{2N-N_f+2}}{(-F)^{N_1}}\right]^{\frac{1}{2N_0-N_f+2}} \tag{5.1}$$

where we use the notation $\text{USp}(2N)\to\widehat{\text{USp}(2N_0)}\times \text{U}(N_1)$ with $2N=2N_0+2N_1$.

As in SO(N) case there is a special case in which the condition $(2N_0-N_f+2)=0$ is satisfied and the matrix model curve is given as

$$y_{m,d}^2 = \left(x^2+D\right)^2-\frac{4\Lambda^4}{\eta^2}\,, \qquad \eta^{2N_1}=1\,, \qquad \text{for } (2N_0-N_f+2)=0\,. \tag{5.2}$$

Let us demonstrate these general features (5.1) and (5.2) explicitly by comparing them with the results from strong-coupling approach obtained in [20] already. Let us consider USp($2N$) gauge theories with $N_f(\leq 2N+1)$ flavors where $N=2$ or 3.

[22]The relation between $W_3'(x)$ and $F_4(x)$ can be written as

$$\begin{aligned} y_{m,d}^2 &= F_4(x)=\left(\frac{W_3'(x)}{x}\right)^2+4F\,, && N_f < 2N+1\,,\\ y_{m,d}^2 &= F_4(x)=\left(\frac{W_3'(x)}{x}\right)^2+4F-4x^2\Lambda^2\,, && N_f = 2N+1\,. \end{aligned}$$

In particular, when $N_f=2N+1$, the matrix model curve $y_{m,d}^2$ has an extra contribution from $4x^2\Lambda^2$. Therefore in this particular case, the curve should be $y_{m,d}^2=\left(x^2-m-2\Lambda^2\right)^2+4F$ and $m+2\Lambda^2=\frac{K^2-F}{K}$.

USp(4) with $N_f = 2$. In [20], the first kind solution belongs to the degenerated case. To see the equivalence we put the values $(2N_0, N_1, N_f) = (0, 2, 2)$ into our formula (5.1) (Note $2N = 2N_0 + 2N_1$). However in this breaking pattern, since we have the relation $(2N_0 - N_f + 2) = 0$, the matrix model curve (5.2) is given by

$$y_{m,d}^2 = \left(x^2 + D\right)^2 - \frac{4\Lambda^4}{\eta^2}, \qquad \eta^4 = 1\,.$$

The factorization problem [20] resulted in $\widehat{y_m}^2 = (x^2 - \alpha^2)^2 \mp 4\Lambda^4$. The gauge group breaks into $\mathrm{USp}(4) \to \widehat{\mathrm{USp}(0)} \times \mathrm{U}(2)$. One can see the equivalence between two approaches by identifying $\alpha^2 = -D$.

USp(4) with $N_f = 3$. The first kind of solution in [20] is the one corresponding to the degenerated case with the breaking pattern $\widehat{y_m}^2 = (t + D)^2 + 4F$, $D = \frac{F^2}{\Lambda^6} - \frac{\Lambda^6}{F}$. The $-\alpha^2$ corresponds to D here. To see the agreement, we put the values $(2N_0, N_1, N_f) = (2, 1, 3)$ into (5.1) (the breaking pattern with $(N_0, N_1, N_f) = (0, 2, 3)$ where $\mathrm{USp}(4) \to \widehat{\mathrm{USp}(0)} \times \mathrm{U}(2)$ reproduces the same result) and get the matrix model curve,

$$y_{m,d}^2 = (x^2 - m)^2 + 4F\,, \qquad m = \frac{\Lambda^6}{F} - \frac{F^2}{\Lambda^6}\,.$$

Therefore, there is an agreement.

USp(4) with $N_f = 4$.

1. The first kind of solution in [20] is represented as $\widehat{y_m}^2 = (t + d)^2 + 4\eta\Lambda^2 t$, $\eta^2 = 1$ and the breaking pattern is $\widehat{\mathrm{USp}(0)} \times \mathrm{U}(2)$. The α^2 corresponds to $-d \pm 2\Lambda^2$. If we change the parametrization for matrix model curve by $d = \frac{-F}{\eta\Lambda^2} - \eta\Lambda^2$, it is rewritten as $\widehat{y_m}^2 = (t + \eta\Lambda^2 - \frac{F}{\eta\Lambda^2})^2 + 4F$. To see the agreement, we put the values $(2N_0, N_1, N_f) = (0, 2, 4)$ into our general formula (5.1) and get the matrix model curve as
$$y_{m,d}^2 = (x^2 - m)^2 + 4F\,, \qquad m = \frac{\eta F}{\Lambda^2} - \eta\Lambda^2$$
where $\eta^2 = 1$. We see an agreement.

2. For the other breaking pattern $\mathrm{USp}(4) \to \widehat{\mathrm{USp}(2)} \times \mathrm{U}(1)$, we check the equivalence by putting the values $(2N_0, N_1, N_f) = (2, 1, 4)$ into (5.1). In this case since the condition $(2N_0 - N_f + 2) = 0$ is satisfied, the matrix model curve (5.2) is given by
$$y_{m,d}^2 = (x^2 + D)^2 - 4\Lambda^4\,.$$
Since the results from the factorization problem is $\widehat{y_m}^2 = \left(t - \alpha^2\right)^2 - 4\Lambda^4$, we see a perfect agreement.

USp(4) **with** $N_f = 5$.

1. For the first kind of solution, the matrix model curve can be written as $\widehat{y_m}^2 = (t+D)^2 + 4F, D = \epsilon\left(\frac{F^2}{\Lambda^2}\right)^{\frac{1}{3}} - \epsilon F\left(\frac{\Lambda^2}{F^2}\right)^{\frac{1}{3}}$ where $\epsilon^3 = 1$. The D corresponds to $a/2$ in [20]. We use different parametrization and the breaking pattern is $\mathrm{USp}(4) \to \widehat{\mathrm{USp}(0)} \times \mathrm{U}(2)$. Putting the values $(2N_0, N_1, N_f) = (0, 2, 5)$ into (5.1) with the footnote 22 we get the matrix model curve as

$$y_{m,d}^2 = (x^2 - m - 2\Lambda^2)^2 + 4F\,, \qquad m = -\eta\left(\frac{F^2}{\Lambda^2}\right)^{\frac{1}{3}} + \frac{F}{\eta}\left(\frac{\Lambda^2}{F^2}\right)^{\frac{1}{3}} - 2\Lambda^2$$

   with $\eta^3 = 1$. An agreement between two approaches is evident.

2. By putting the values $(2N_0, N_1, N_f) = (2, 1, 5)$ into (5.1) with the footnote 22 where $\mathrm{USp}(4) \to \widehat{\mathrm{USp}(2)} \times \mathrm{U}(1)$, we get

$$y_{m,d}^2 = (x^2 - m - 2\Lambda^2)^2 + 4F\,, \qquad m = \frac{F}{\Lambda^2} - 3\Lambda^2\,.$$

   From the results in [20], the second kind of solution can be represented as $\widehat{y_m}^2 = (t+D)^2 + 4\Lambda^2(3\Lambda^2 + m)$. If we introduce $F = \Lambda^2(3\Lambda^2 + m)$, we get $m = -3\Lambda^2 + \frac{F}{\Lambda^2}$ and see an agreement.

USp(6) **with** $N_f = 2$. The matrix model curve is $\widehat{y_m}^2 = (t - \frac{a}{2})^2 - 4\epsilon\Lambda^4$, where $\epsilon^3 = 1$ and the breaking pattern is $\mathrm{USp}(6) \to \widehat{\mathrm{USp}(0)} \times \mathrm{U}(3)$. In this case since the condition $(2N_0 - N_f + 2) = 0$ is satisfied, the matrix model curve (5.2) from glueball approach is given as

$$y_{m,d}^2 = \left(x^2 + D\right)^2 - \frac{4\Lambda^4}{\eta^2}\,, \qquad \eta^6 = 1\,.$$

Therefore, by identifying $(\frac{-a}{2}, \epsilon)$ with (D, η^4), we can see the agreement between two results.

USp(6) **with** $N_f = 4$.

1. For the first kind of the solution from the factorization problem it was given in [20] as $\widehat{y_m}^2 = (t+D)^2 + 4F$, $D = \eta\frac{F^2 + \Lambda^8}{\sqrt{F}\Lambda^4}$, where $\eta^4 = 1$. We use different parametrization to see the equivalence easily. By putting the values $(2N_0, N_1, N_f) = (4, 1, 4)$ where $\mathrm{USp}(6) \to \widehat{\mathrm{USp}(4)} \times \mathrm{U}(1)$ (We also get the same results if we put $(2N_0, N_1, N_f) = (0, 3, 4)$. Therefore, we cannot tell which one corresponds to $(\frac{1}{2\pi i}\oint_{A_0} T(x)dx,\ \frac{1}{2\pi i} \times \oint_{A_1} T(x)dx)$ on this branch without doing these computations), we obtain the matrix model curve as

$$y_{m,d}^2 = \left(x^2 - m\right)^2 + 4F, \qquad m = \frac{\eta\Lambda^4}{\sqrt{F}} - \eta^3\frac{F\sqrt{F}}{\Lambda^4}$$

   where $\eta^4 = 1$. Therefore, the results obtained from two approaches are the same.

JHEP02(2004)009

2. For the second solution, the matrix model curve is $\widehat{y_m}^2 = \widehat{(t+d)^2 - 4\epsilon\Lambda^4}$, where $\epsilon^2 = 1$ and the breaking pattern is given by $\mathrm{USp}(6) \to \widehat{\mathrm{USp}(2)} \times \mathrm{U}(2)$. In this case since the condition $(2N_0 - N_f + 2) = 0$ is satisfied, the matrix model curve (5.2) from glueball approach is given as

$$y_{m,d}^2 = \left(x^2 + D\right)^2 - \frac{4\Lambda^4}{\eta^2}\,, \qquad \eta^4 = 1\,.$$

Therefore, by identifying (d, ϵ) with (D, η^2), we can see the agreement between two results. This solution is exactly the same as the degenerated solution with $\mathrm{USp}(4) \to \widehat{\mathrm{USp}(0)} \times \mathrm{U}(2)$ with $N_f = 2$: We can expect to have this feature from the addition map.

$\mathrm{USp}(6)$ **with** $N_f = 6$**.**

1. The matrix model curve is $\widehat{y_m}^2 = \widehat{(t+d)^2} - 4\epsilon\Lambda^2 t$, where $\epsilon^2 = 1$ and the breaking pattern is given by $\mathrm{USp}(6) \to \widehat{\mathrm{USp}(2)} \times \mathrm{U}(2)$. In this case by putting the values $(2N_0, N_1, N_f) = (2, 2, 6)$ into (5.1), the matrix model curve from glueball approach is given by

$$y_{m,d}^2 = \left(x^2 - m\right)^2 + 4F\,, \qquad m = \frac{\eta F}{\Lambda^2} - \eta\Lambda^2$$

where η is 2-nd root of unity. Therefore, by identifying $F = d\epsilon\Lambda^2 - \Lambda^4$ and $\eta = \epsilon$ respectively we can see the agreement between two results. These solutions are exactly the same as the degenerated solution with $\mathrm{USp}(4) \to \widehat{\mathrm{USp}(0)} \times \mathrm{U}(2)$ we have discussed ($N_f = 4$): We can expect to have this feature from the addition map.

2. For the breaking pattern $\mathrm{USp}(6) \to \widehat{\mathrm{USp}(4)} \times \mathrm{U}(1)$, the condition $(2N_0 - N_f + 2) = 0$ is satisfied and the matrix model curve is given by

$$y_{m,d}^2 = \left(x^2 + D\right)^2 - 4\Lambda^4$$

which agrees with the one in [20]. The D here corresponds to $-s_1$ there. We can see this solution from the breaking pattern $\mathrm{USp}(4) \to \widehat{\mathrm{USp}(2)} \times \mathrm{U}(1)$ through the addition map.[23]

In summary, for the $\mathrm{USp}(2N)$ gauge theories with $N_f(\leq 2N+1)$ flavors where $N = 2, 3$, the solutions (5.1) and (5.2) coincide with the matrix model curve from the strong-coupling approach except that we could not check the $\mathrm{USp}(6)$ gauge theory with $N_f = 6$ where there exists a breaking pattern $\mathrm{USp}(6) \to \widehat{\mathrm{USp}(0)} \times \mathrm{U}(3)$ (in this case, $N_0 = 0$). Since the SW curve for $\mathrm{USp}(2N) \to \mathrm{USp}(0) \times \mathrm{U}(N)$ has less power of x^2 compared with the SW curve for $\mathrm{USp}(2N) \to \widehat{\mathrm{USp}(2N_0)} \times \mathrm{U}(N_1), N_0 \neq 0$, the factorization problem will be more complicated due to the existence of many parameters. Therefore, one expects that our matrix model curve (5.1) and (5.2) will predict the solutions for the degenerated $\mathrm{USp}(2N)$ gauge theories with $N_f(\leq 2N+1)$ massless flavors where $N \geq 4$ and $\mathrm{USp}(2N) \to \widehat{\mathrm{USp}(2N_0)} \times \mathrm{U}(N_1), N_0 \neq 0$.

[23] For the last breaking pattern $\mathrm{USp}(6) \to \widehat{\mathrm{USp}(0)} \times \mathrm{U}(3)$, since the factorization problem was not solved exactly in [20], we cannot check the results explicitly. We simply put the results from glueball approach, $y_{m,d}^2 = (x^2 + m)^2 + 4F$, and $m = \frac{\eta F^{\frac{3}{4}}}{\Lambda} - \frac{F^{\frac{1}{4}}\Lambda}{\eta}, \eta^8 = 1$.

JHEP02(2004)009

6. The curve for degenerated $\mathrm{SO}(N)$ gauge theory with massive flavors

For the massive case, there exists one extra condition $W_3'(\pm m_f) = 0$ where $m_f(\neq 0)$ is the mass of flavor which is different from the m introduced in [15] or section 2 as a parameter of a matrix model curve. From this condition, the mass of flavor can be written as

$$m_f^2 = -\frac{1}{2}(x_0^2 + x_1^2 + x_2^2), \qquad W_3'(x) = x(x^2 - m_f^2)\,.$$

Equivalently, $m_f = \pm i\sqrt{\frac{x_0^2+x_1^2+x_2^2}{2}}$ is pure imaginary where the matrix model curve is described by $y_m^2 = \prod_{i=0}^{2}(x^2 + x_i^2)$ with $x_0 < x_1 < x_2$. The fact that m_f is pure imaginary implies the flavor-dependent part $\mathcal{F}_{\text{flavor}}$ can be written in terms of the dual period Π_1 (not Π_0) and other term. Therefore, we encounter a complete different story when the masses of flavors are nonzero. The absolute value $|m_f|$ is greater than x_0: $x_0 < |m_f|$. When we take the singular limit $x_1^2 \to x_0^2$, the three-branch cuts $[-ix_2, -ix_1], [-ix_0, ix_0]$ and $[ix_1, ix_2]$ are joined at $x = \pm ix_0$.

As in massless cases, we want to study two topics, 1) the behavior of $n = 0$ and $n = 1$ singularities and 2) the curve for degenerated case. Although for the degenerated case, we obtain a general matrix model curve (in the sense that the curve holds for the generic value of N), as we will see below, there exist some difficulties for $n = 0$ and $n = 1$ singularities which will be discussed in next section. Thus, in this section we concentrate on the degenerated case in which the matrix model curve can be written as $y_m^2 = x^2(x^2 + a^2)(x^2 + b^2)$. According to the notation above, we simply put the values (x_0, x_1, x_2) as $(0, a, b)$. There are two-branch cuts $[-ib, -ia]$ and $[ia, ib]$. The mass of flavors can be written as $m_f = \pm i\sqrt{\frac{a^2+b^2}{2}}$, as before. Therefore, the position of mass is always on the branch cuts: $a < |m_f| < b$. Recall that the property of the integral,[24]

$$\int_{ix_0}^{\Lambda_0} y_m dx = \int_{P}^{\Lambda_0} y_m dx = \int_{ix_2}^{\Lambda_0} y_m dx\,, \qquad ix_0 < P < ix_2\,.$$

In addition, we have to pay attention to the form of $\mathcal{F}_{\text{flavor}}$. Naively the contributions from $\mathcal{F}_{\text{flavor}}$ is given by (2.1). However, to get the phase factor correctly it should be as follows and by simple manipulations it is written as a dual period Π_1 plus other term:

$$\mathcal{F}_{\text{flavor}} = N_f \int_{m_f}^{\Lambda_0} y_m dx + N_f \int_{-\Lambda_0}^{-m_f} y_m dx = 2\pi i N_f \Pi_1 + 2N_f \int_{i\Lambda_0}^{\Lambda_0} y_m dx\,.$$

[24]One can consider a noncompact cycle $\widetilde{B}_1$ which has an intersection number one with A_1 compact cycle and meet at $x = m_f = P$. Then since the closed loop $B_1 - \widetilde{B}_1$ does not contain any pole in the $y_m(x)$, there exists a relation $\oint_{B_1 - \widetilde{B}_1} y_m dx = 0$. Therefore, the dual period $\Pi_1 = \frac{1}{2\pi i}\oint_{B_1} y_m dx$ can be written as $\frac{1}{2\pi i}\int_{ix_2}^{i\Lambda_0} y_m dx + \frac{1}{2\pi i}\int_{i\Lambda_0}^{ix_2}(-y_m)dx$ which will be $\frac{1}{\pi i}\int_{ix_2}^{i\Lambda_0} y_m dx$. We perform similar computation for the cycle $\widetilde{B}_1$ and obtain $\frac{1}{\pi i}\int_{P}^{i\Lambda_0} y_m dx$. By changing the upper bound $i\Lambda_0$ into Λ_0 with an extra term, we get the relation $\int_P^{\Lambda_0} y_m dx = \int_{ix_2}^{\Lambda_0} y_m dx$.

Note that we used the fact that $\int_{ix_2}^{i\Lambda_0} y_m dx = \pi i \Pi_1$. In previous paper [15], there was a typo in a dual period Π_1. Therefore, the effective superpotential with massive flavors in degenerated branch can be given as

$$W_{\rm eff} = 2\pi i[(N_0-2)\Pi_0+(2N_1-N_f)\Pi_1]-2(N-2-N_f)S\log\left(\frac{\Lambda}{\Lambda_0}\right)-2N_f\int_{i\Lambda_0}^{\Lambda_0} y_m dx\,. \quad (6.1)$$

It is ready to calculate the equations of motion for the fields f_0 and f_2, in general. When we compute the equation of motion for a field F for degenerated case, it is noteworthy to observe the following relation under the large Λ_0 limit where $y_m^2 = x^2(x^2+a^2)(x^2+b^2) = x^2[(x^2-m_f^2)^2+4F]$,

$$2N_f\int_{i\Lambda_0}^{\Lambda_0}\frac{\partial y_m}{\partial F}dx \simeq -4N_f\log i.$$

By using previous results given in [15][25] and this $\log i$-term, we obtain one equation of motion,

$$(N_0-2)\log\left|\frac{4\Lambda^2}{(a+b)^2}\right| + (2N_1-N_f)\log\left|\frac{4\Lambda^2}{(b^2-a^2)}\right| - N_f\log i = 0\,. \quad (6.2)$$

Note the last term for the phase factor which comes from the last term in the formula (6.1). Strictly speaking, the coefficient in the last term (6.2) we got from the glueball approach is 2, not -1. It is not clear how this arises but on the other hand, the strong-coupling approach implies that the relation (6.2) should be correct. Therefore, after we solve this equation, the matrix model curve[26] can be represented as

$$y_{m,d}^2 = (x^2-m_f^2)^2+4F\,, \qquad m_f^2 = -\frac{K^2-F}{K}\,, \qquad K \equiv \left[\frac{(\epsilon\Lambda^2)^{N-2}(\epsilon i\Lambda^2)^{-N_f}}{(\eta i\sqrt{F})^{2N_1-N_f}}\right]^{\frac{1}{N_0-2}}. \quad (6.3)$$

where $\epsilon^2 = \eta^2 = 1$. Recall that $m_f^2 = -(a^2+b^2)/2$ and $4F = -(b^2-a^2)^2/4$. The quartic (in x) matrix model curve in the breaking pattern $\mathrm{SO}(N) \to \mathrm{SO}(N_0)\times\widehat{\mathrm{U}(N_1)}$ with N_f massive flavors ($N = N_0+2N_1$) depends on N_0, N_1, N_f and a field F. Contrary to the non-degenerated case, this formula is always valid (On the non-degenerated case, there exists another condition $c<0$ where c is given by (7.1) or (7.2) which will be discussed in next section). Note that we use a different notation to see the condition $W_3'(\pm m_f) = 0$ easily. When the condition $(N_0-2)=0$ is satisfied, we have to use different formula, as we did before,

$$y_{m,d}^2 = (x^2-m_f^2)^2 - \frac{4\Lambda^4}{\eta^2}\,, \qquad \eta^{2N_1-N_f} = i^{N_f}\,, \qquad \text{for } (N_0-2)=0\,. \quad (6.4)$$

Here we used the equation (6.2) when $(N_0-2)=0$ and identified η with $\pm\frac{4\Lambda^2}{b^2-a^2}$. Note $m_f^2 = -\frac{1}{2}(a^2+b^2)$.

[25] There are two relations for the derivatives of dual periods with respect to a field F: $2\pi i\frac{\partial\Pi_0}{\partial F} = 2\log\left|\frac{4\Lambda^2}{(a+b)^2}\right|$, and $2\pi i\frac{\partial\Pi_1}{\partial F} = 2\log\left|\frac{4\Lambda^2}{(b^2-a^2)}\right|$.

[26] In particular, when $N_f = N-3$, the matrix model curve $y_{m,d}^2$ has an extra contribution from $4\Lambda^2(x^2-m^2)$. Therefore, in this particular case, the curve should be $y_{m,d}^2 = (x^2-m_f^2-2\Lambda^2)^2+4F$ and $m_f^2+2\Lambda^2 = -\frac{K^2-F}{K}$.

The discussions above are valid to the $r=0$ branch studied in [19]. The immediate question is how do we represent the matrix model curve for $r\neq 0$ vacua? As already discussed in [6], the nonzero r-vacua can be realized by changing a mass parameter m_f. If the singularity passes through the first cut (enclosed by the A_1 contour), the A_1 contour is deformed and has transformed to $A_1+C_1-\widetilde{C}_1$. Note that in our present case, the contours A_1 and $C_1(\widetilde{C}_1)$ correspond to a contour around the branch cut $[ix_1, ix_2]$ and a contour around m_f on the upper(lower) sheet respectively. On the pseudo-confining vacua the singularity is located on the second sheet, so we have $\frac{1}{2\pi i}\oint_{\widetilde{C}_1} T(z)dz=0$ and $\frac{1}{2\pi i}\oint_{C_1} T(z)dz=1$. After transition we have $N_1-1=\frac{1}{2\pi i}\oint_{A_1} T(z)dz$. Thus, the effective value of N_1 is reduced by 1 in this transition. Therefore, if we pass through r-singularities, after transition the effective value of N_1 has been reduced by (N_1-r). In addition, the effective number of flavor is also reduced by (N_f-r). The contribution from the singularity on the first sheet has different sign from the one on the second sheet because the sign of y_m is different between the two sheets. Therefore, by adding the quantity r from the second sheet, $\mathcal{F}_{\text{flavor}}$ on the r-branch can be summed and represented as follows:

$$\begin{aligned}\mathcal{F}_{\text{flavor}} &= (N_f-r)\int_{m_f}^{\Lambda_0} y_m dx + (N_f-r)\int_{-\Lambda_0}^{-m_f} y_m dx + r\int_{m_f}^{\Lambda_0}(-y_m)dx + r\int_{-\Lambda_0}^{-m_f}(-y_m)dx \\ &= 2\widetilde{N_f}\int_{ix_2}^{\Lambda_0} y_m dx = 2\pi i\widetilde{N_f}\Pi_1 + 2\widetilde{N_f}\int_{i\Lambda_0}^{\Lambda_0} y_m dx\end{aligned}$$

where we introduce the following notation,

$$\widetilde{N}=N-2r\,, \qquad \widetilde{N_f}=N_f-2r\,, \qquad \widetilde{N_1}=N_1-r\,. \tag{6.5}$$

Therefore, by combining the modified N_1 and the contribution from the flavors, after transition the effective superpotential can be given as

$$W_{\text{eff}}=2\pi i[(N_0-2)\Pi_0+(2\widetilde{N_1}-\widetilde{N_f})\Pi_1]-2(\widetilde{N}-2-\widetilde{N_f})S\log\left(\frac{\Lambda}{\Lambda_0}\right)-2\widetilde{N_f}\int_{i\Lambda_0}^{\Lambda_0} y_m dx\,. \tag{6.6}$$

In the log term, we used the fact $\widetilde{N}-2-\widetilde{N_f}=N-2-N_f$. This effective superpotential for nonzero r-vacua takes the same form for $r=0$ vacua (6.1) with modified quantities (6.5) and therefore includes (6.1).

Now it is straightforward to obtain the matrix model curve for $r\neq 0$ vacua. The general matrix model curve[27] can be given by starting with (6.3) and changing the values (N, N_1, N_f) appearing in (6.3) into $(\widetilde{N}, \widetilde{N_1}, \widetilde{N_f})$ where they are defined as in (6.5):

$$y_{m,d}^2=(x^2-m_f^2)^2+4F\,, \qquad m_f^2=-\frac{K^2-F}{K}\,, \qquad K\equiv\left[\frac{(\epsilon\Lambda^2)^{\widetilde{N}-2}(\epsilon i\Lambda^2)^{-\widetilde{N_f}}}{(\eta i\sqrt{F})^{2\widetilde{N_1}-\widetilde{N_f}}}\right]^{\frac{1}{N_0-2}} \tag{6.7}$$

where $\epsilon^2=\eta^2=1$ and moreover when the N_0 is equal to 2, as we did before, the matrix model curve with modified quantities (6.5) becomes

$$y_{m,d}^2=(x^2-m_f^2)^2-\frac{4\Lambda^4}{\eta^2}\,, \qquad \eta^{2\widetilde{N_1}-\widetilde{N_f}}=i^{\widetilde{N_f}}\,, \qquad \text{for } (N_0-2)=0\,. \tag{6.8}$$

[27] When $\widetilde{N_f}=\widetilde{N}-3$, the matrix model curve should be $y_{m,d}^2=\left(x^2-m_f^2-2\Lambda^2\right)^2+4F$ and $m_f^2+2\Lambda^2=-\frac{K^2-F}{K}$.

Therefore both equations (6.7) and (6.8) are the most general expressions including (6.3) and (6.4). To demonstrate these general solutions, let us consider some explicit examples, $\mathrm{SO}(N)$ theories with $N_f(\leq N-3)$ flavors for $N=4,5$, or 6. For SO(6) gauge theories, we only consider even number of flavors as we did in [19].

SO(4) with $N_f = 1$: non-baryonic $r = 0$ branch. $\mathrm{SO}(4) \to \mathrm{SO}(2) \times \widehat{\mathrm{U}(1)}$. The matrix model curve derived by the factorization problem of degenerated case was given in [19]. After the factorization (Note that the $F_6(x)$ has an overall factor x^2 in the non-degenerated case) we have obtained $F_4(x) = (x^2 - A)^2 - 4\Lambda^2(x^2 - m_f^2)$ and from the relation between $W_3'(x)$ and $F_4(x)$ one can read off the $W_3'(x)$. By the condition $W_3'(\pm m_f) = 0$, we see $A = m_f^2$ and the matrix model curve from strong-coupling approach can be written as $\widehat{y_m}^2 = (x^2 - m_f^2 - 2\Lambda^2)^2 - 4\Lambda^4$.

Now let us see this observation from the glueball approach. If we substitute the values $(N_0, N_1, N_f) = (2,1,1)$ where $N = N_0 + 2N_1$ into our general formula (6.4) (In this case the condition $(N_0 - 2) = 0$ is satisfied and also see the footnote 26), we obtain the matrix model curve on the degenerated case,

$$y_{m,d}^2 = (x^2 - m_f^2 - 2\Lambda^2)^2 + 4\Lambda^4$$

with the phase factor $\eta = i$. Therefore, we do not see the agreement:$\widehat{y_m}^2 \neq y_{m,d}^2$. Let us go back a derivation for (6.4). According to the strong-coupling approach, the curve is characterized by $\widehat{y_m}^2 = (x^2 - m_f^2)(x^2 - m_f^2 - 4\Lambda^2)$. Then a^2 in the matrix model curve $y_{m,d}^2 = (x^2 + a^2)(x^2 + b^2)$ corresponds to $-m_f^2 - 4\Lambda^2$ and b^2 corresponds to $-m_f^2$. So in this particular case, the previous condition $a < |m_f| < b$ does not hold. In other words, the glueball approach is not allowed.[28]

SO(5) with $N_f = 1$: non-baryonic $r = 0$ branch. To see the equivalence between two approaches easily, we solve the factorization again by using the following parametrization, $P_4^2(x) - 4x^2\Lambda^4(x^2 - m_f^2) = H_2^2(x)[(x^2 - b)^2 + 4F]$. Note $4F = c$ in the notation of [19]. From the relationship between $F_4(x)$ and $W_3'(x)$, we obtain $b = m_f^2$ and additionally there exists one equation corresponding to (5.15) of [19]. However for the present purpose, we reexpress it in terms of F by writing a as a function of F or c, $F^4 - 2F^2\Lambda^8 + \Lambda^{16} - F\Lambda^8 m_f^4 = 0$. Solving this equation, we can get m_f^2 as follows: $m_f^2 = \pm\frac{F^2 - \Lambda^8}{\sqrt{F}\Lambda^4}$. In this case, the matrix model curve is written as $\widehat{y_m}^2 = \left(x^2 - m_f^2\right)^2 + 4F$.

[28] The first kind of solution in [19] where $\mathrm{SO}(4) \to \widehat{\mathrm{U}(2)}$ has the matrix model curve as $\widehat{y_m}^2 = (x^2 - D)^2 + 4F$ where $D = -b/2$ and $4F = c - b^2/4$ in the notation of [19]. In order to get these results, we used two relations: One for c and the other is $a^2(a + 9\Lambda^2) + 3a(a + 4\Lambda^2)m_f^2 + (3a + 4\Lambda^2)m_f^4 + m_f^6 = 0$ together with $b = -2m_f^2 - 4\Lambda^2$. By using these two relations, one can get an equation for m_f^2 or D in terms of F or c. To see the equivalence between two approaches (glueball approach and strong-coupling approach), we put $(N_0, N_1, N_f) = (0,2,1)$ into the general formula (6.3) and get the matrix model curve $y_{m,d}^2 = (x^2 - m_f^2 - 2\Lambda^2)^2 + 4F$, and $m_f^2 = -\frac{\eta F^{\frac{3}{4}}}{\Lambda} + F\frac{\Lambda}{\eta F^{\frac{3}{4}}} + 2\Lambda^2$ where η is 4-th root of unity. This leads to a disagreement.

On the other hand, if we put the values $(N_0, N_1, N_f) = (3,1,1)$ where $\mathrm{SO}(5) \to \mathrm{SO}(3) \times \widehat{\mathrm{U}(1)}$ into (6.3) we obtain the matrix model curve,

$$y_{m,d}^2 = (x^2 - m_f^2)^2 + 4F\,, \qquad m_f^2 = \pm\frac{(\Lambda^8 - F^2)}{\Lambda^4\sqrt{F}}\,.$$

Therefore, there is a perfect agreement. Although there is a breaking pattern $\mathrm{SO}(5) \to \mathrm{SO}(1) \times \widehat{\mathrm{U}(2)}$, the corresponding matrix model curve does not exist.

SO(5) with $N_f = 2$: non-baryonic $r = 1$ branch. From the solution for the factorization, we have obtained $\widehat{y_m}^2 = (x^2 - s)^2 - 4x^2\Lambda^2$ where we wrote the characteristic function $P_2(x)$ in [19] as $(x^2 - s)$. Taking into account the condition $W_3'(\pm m_f) = 0$, we get $m_f^2 = s$. If we introduce a new notation $4F \equiv -4\Lambda^4 - 4s\Lambda^2$, the mass of flavor can be rewritten as $m_f^2 = -\Lambda^2 - \frac{F}{\Lambda^2}$ and the curve becomes $\widehat{y_m}^2 = (x^2 - m_f^2 - 2\Lambda^2)^2 + 4F$. Since the breaking pattern of this solution is $\mathrm{SO}(5) \to \mathrm{SO}(1) \times \widehat{\mathrm{U}(2)}$, by putting $(\widetilde{N}, \widetilde{N_1}, \widetilde{N_f}, N_0) = (3,1,0,1)$ into our general formula (6.7) (we took $\epsilon = -1$) with the footnote 27, we obtain the matrix model curve,

$$y_{m,d}^2 = (x^2 - m_f^2 - 2\Lambda^2)^2 + 4F\,, \qquad m_f^2 = -\Lambda^2 - \frac{F}{\Lambda^2}\,.$$

Therefore, we see the agreement between two approaches.

SO(5) with $N_f = 2$: non-baryonic $r = 0$ branch. By using the same parametrization as [19] with $F_4(x) = (x^4 + 4bx^2 + 2c)$ we can represent c as a function of a, b and m_f and there is an extra equation characterized by, $a^2 + m_f^4 + 2a(2\Lambda^2 + m_f^2) = 0$. To see the equivalence between two approaches let us introduce a new notation $4F \equiv c - 4b^2$ and take into account the condition $W_3'(\pm m_f) = 0$. Finally, by taking $2b = -m_f^2 - 2\Lambda^2$ and solving the two equations (the relation for c and the above equation) we get the mass $m_f^2 = -3\Lambda^2 + \frac{F}{\Lambda^2}$. The matrix model curve can be written as $\widehat{y_m}^2 = (x^2 - m_f^2 - 2\Lambda^2)^2 + 4F$. If we put $(N_0, N_1, N_f) = (1,2,2)$ where there is $\mathrm{SO}(5) \to \mathrm{SO}(1) \times \widehat{\mathrm{U}(2)}$ into (6.3) (we took $\epsilon = -1$) with the footnote 26, we obtain the matrix model curve

$$y_{m,d}^2 = (x^2 - m_f^2 - 2\Lambda^2)^2 + 4F\,, \qquad m_f^2 = \frac{F}{\Lambda^2} - 3\Lambda^2\,.$$

We see an agreement. In this case, for $(m_f^2 + 2\Lambda^2)^2 + 4F = 0$, there is a Chebyshev branch with $\mathrm{SO}(5) \to \mathrm{SO}(5)$.

SO(6) with $N_f = 1$: non-baryonic $r = 0$ branch. If we rewrite c in the results of [19] as $c = 4F$ the matrix model curve for degenerated case can be represented as $\widehat{y_m}^2 = (x^2 - m_f^2)^2 + 4F$ with one equation, $F^6 - 2F^3\Lambda^{12} - 4F^2 m_f^4 \Lambda^{12} - F m_f^8 \Lambda^{12} + \Lambda^{24} = 0$ which is the last equation of (5.18) in [19]. Solving this constraint, we can obtain m_f^2 as follows: $m_f^2 = \pm\frac{(F^{\frac{3}{2}} + \Lambda^6)}{i\Lambda^3 F^{\frac{1}{4}}}$, or $\pm\frac{(F^{\frac{3}{2}} - \Lambda^6)}{\Lambda^3 F^{\frac{1}{4}}}$.

To see the agreement with our general formula we simply put $(N_0, N_1, N_f) = (4,1,1)$ where there exists $\mathrm{SO}(6) \to \mathrm{SO}(4) \times \widehat{\mathrm{U}(1)}$ into (6.3) we can get the matrix model curve

$$y_{m,d}^2 = (x^2 - m_f^2)^2 + 4F\,, \qquad m_f^2 = \pm\frac{(\Lambda^6 - F^{\frac{3}{2}})}{F^{\frac{1}{4}}\Lambda^3}\,, \qquad \pm i\frac{(\Lambda^6 + F^{\frac{3}{2}})}{F^{\frac{1}{4}}\Lambda^3}\,.$$

This implies a perfect agreement. For $m_f^4 + 4F = 0$, a Chebyshev branch $\mathrm{SO}(6) \to \mathrm{SO}(6)$ appears.

$\mathrm{SO}(6)$ **with** $N_f = 2$**: non-baryonic** $r = 0$ **branch.** There are two types of solutions in [19], one has a t factor in $H_2(t)$ which can be written as $tH_1(t)$ and the other has no t factor in $H_2(t)$.

1. $\mathrm{SO}(6) \to \mathrm{SO}(2) \times \widehat{\mathrm{U}(2)}$. First, let us consider the former case. The matrix model curve given in [19] is $\widehat{y_m}^2 = \left(t - m_f^2\right)^2 + 4\Lambda^4$ by inserting $a = m_f^2 + 2\eta\Lambda^2$ into $F_4(x)$. By putting $(N_0, N_1, N_f) = (2, 2, 2)$ into our general formula (6.4) we obtain the matrix model curve from the viewpoint of glueball approach,
$$y_{m,d}^2 = (x^2 - D)^2 + 4\Lambda^4 \,.$$
This leads to an agreement between two approaches. When $D^2 + 4\Lambda^4 = 0$, there is a Chebyshev branch $\mathrm{SO}(6) \to \mathrm{SO}(6)$.

2. $\mathrm{SO}(6) \to \widehat{\mathrm{U}(3)}$. Secondly, we move to the latter case. The solutions were displayed in equation (5.22) of [19]. If we use a new notation $F \equiv -\left(b + \Lambda^2\right)^2$, the matrix model curve can be rewritten as $\widehat{y_m}^2 = \left(x^2 - m_f^2\right)^2 + 4F$ with $m_f^2 = \pm i\frac{(F+\Lambda^4)}{\Lambda^2}$. To see the equivalence with our general formula, we put the breaking pattern on this solution, namely $\mathrm{SO}(6) \to \widehat{\mathrm{U}(3)}$ into the formula (6.7) and get the matrix model curve
$$y_{m,d}^2 = \left(x^2 - m_f^2\right)^2 + 4F \,, \qquad m_f^2 = \pm i\frac{(F + \Lambda^4)}{\Lambda^2} \,.$$
One can see an agreement between two approaches.

$\mathrm{SO}(6)$ **with** $N_f = 2$**: non-baryonic** $r = 1$ **branch.**

1. $\mathrm{SO}(6) \to \widehat{\mathrm{U}(3)}$. The first kind of solution studied in [19] gives a relation for mass $m_f^2 = a - 2\eta\Lambda^2$. There was a typo in the equation of [19]. If we introduce a new notation $4F \equiv 4a\eta\Lambda^2 - 4\Lambda^4$, the mass can be rewritten as $m_f^2 = \eta(\frac{F}{\Lambda^2} - \Lambda^2)$. Since this solution has the breaking pattern $\mathrm{SO}(6) \to \widehat{\mathrm{U}(3)}$ by putting $(\widetilde{N}, \widetilde{N_1}, \widetilde{N_f}, N_0) = (4, 2, 0, 0)$ into our general formula (6.7) we obtain the matrix model curve
$$y_{m,d}^2 = (x^2 - m_f^2)^2 + 4F \,, \qquad m_f^2 = -\epsilon\left(\frac{F}{\Lambda^2} - \Lambda^2\right)$$
where ϵ is 2-th root of unity. This leads to an agreement.

2. $\mathrm{SO}(6) \to \mathrm{SO}(2) \times \widehat{\mathrm{U}(2)}$. The other solution has the matrix model curve $\widehat{y_m}^2 = (x^2 - m_f^2)^2 - 4\Lambda^4$. Since the condition $(N_0 - 2) = 0$ is satisfied, the matrix model curve from (6.8) can be rewritten as
$$y_{m,d}^2 = \left(x^2 - m_f^2\right)^2 - 4\frac{\Lambda^4}{\eta^2} \,, \qquad \eta^2 = 1 \,.$$
A perfect agreement occurs.

SO(6) with $N_f = 3$: non-baryonic $r = 0$ branch. The solution for the breaking pattern $\mathrm{SO}(2) \times \widehat{\mathrm{U}(2)}$ can be represented as follows: $\widehat{y_m}^2 = (x^2 - m_f^2 - 2\Lambda^2)^2 - 4\Lambda^4$. We use $F = \Lambda^2(D + m_f^2 + \Lambda^2)$. From the relationship between $F_4(x)$ and $W_3'(x)$, we have $D = -m_f^2 - 2\Lambda^2$ implying $F = -\Lambda^4$. Putting this relation we get the above matrix model curve.

On the other hand, by substituting the values $(N_0, N_1, N_f) = (2, 2, 3)$ where $\mathrm{SO}(6) \to \mathrm{SO}(2) \times \widehat{\mathrm{U}(2)}$ into (6.4) we obtain the matrix model curve as

$$y_{m,d}^2 = \left(x^2 + D\right)^2 + 4\Lambda^4 .$$

However, as we have discussed in SO(4) with $N_f = 1$ case, the glueball approach given by (6.4) does not include this case also: The location of the flavors is different from the region belonging to the branch cut. For $D^2 - 4\Lambda^4 = 0$, $\mathrm{SO}(6) \to \mathrm{SO}(6)$ Chebyshev branch occurs.

SO(6) with $N_f = 3$: non-baryonic $r = 1$ branch. In [19], it was discussed by using the addition map that this branch was the same as the one for SO(4) with $N_f = 1$ case. Since our general formula also has the same structure, the glueball approach here reproduces the same results from the factorization problem there. That is, for the breaking pattern, $\mathrm{SO}(6) \to \widehat{\mathrm{U}(3)}$, the values $(\widetilde{N_0}, \widetilde{N_1}, \widetilde{N_f}) = (0, 2, 1)$ correspond to $(N_0, N_1, N_f) = (0, 2, 1)$ for SO(4) theory with $N_f = 1$. Moreover, on the breaking pattern $\mathrm{SO}(6) \to \mathrm{SO}(2) \times \widehat{\mathrm{U}(2)}$, the values $(\widetilde{N_0}, \widetilde{N_1}, \widetilde{N_f}) = (2, 1, 1)$ correspond to $(N_0, N_1, N_f) = (2, 1, 1)$ of SO(4) theory with $N_f = 1$.

In summary, for the SO(N) gauge theories with $N_f(\leq N-3)$ flavors where $N = 4, 5$ or 6, we have checked that the solutions (6.3), (6.4), (6.7), and (6.8) coincide with the matrix model curve from the strong-coupling approach except that we could not check the SO(4) gauge theory with $N_f = 1$ where there exists a breaking pattern $\mathrm{SO}(4) \to \widehat{\mathrm{U}(2)}$ (in this case, $N_0 = 0$) and SO(6) with $N_f = 3$ where there exists a breaking pattern $\mathrm{SO}(6) \to \widehat{\mathrm{U}(3)}$ ($N_0 = 0$). Therefore, at least one expects that our matrix model curve (6.3), (6.4), (6.7), and (6.8) will predict the solutions for the degenerated SO(N) gauge theories with $N_f(\leq N - 3)$ massive flavors where $N \geq 7$ and $\mathrm{SO}(N) \to \mathrm{SO}(N_0) \times \widehat{\mathrm{U}(N_1)}, N_0 \neq 0$.[29]

7. Discussion on the $n = 0$ and $n = 1$ singularity with massive flavors

To obtain a general matrix model curve for $n = 0$ and $n = 1$ singularities for our gauge theories, we have to describe the behavior of the matrix model curve $y_m^2 = \prod_{i=0}^{2}\left(x^2 + x_i^2\right)$ in the limit $x_1 \to x_0$. Contrary to the degenerated case we have studied in previous section,

[29]Using this solution from the glueball approach, one obtains the matrix model curve for arbitrary N. For example, for SO(7) with $N_f = 1$ (non-baryonic $r = 0$ branch), when we consider the breaking pattern $\mathrm{SO}(3) \times \widehat{\mathrm{U}(2)}$, by putting $(N_0, N_1, N_f) = (3, 2, 1)$ into our general formula (6.3) we predict the matrix model curve as, $y_{m,d}^2 = (x^2 - m_f^2)^2 + 4F$, and $m_f^2 = \pm\frac{(\Lambda^{16} - F^4)}{F\sqrt{F}\Lambda^8}$. For SO(7) with $N_f = 2$ (non-baryonic $r = 1$ branch), after we put $(\widetilde{N}, \widetilde{N_1}, \widetilde{N_f}, N_0) = (5, 1, 0, 3)$ into (6.7) we obtain the matrix model curve, $y_{m,d}^2 = (x^2 - m_f^2)^2 + 4F$, and $m_f^2 = -\frac{\Lambda^6}{F} + \frac{F^2}{\Lambda^6}$ (In this case, we took $\epsilon = -1$). Moreover, for SO(7) with $N_f = 3$ (non-baryonic $r = 1$ branch), the factorization problem becomes the same as the one for SO(5) with $N_f = 1$ case through the addition map.

there exist two possibilities for the position of the mass of flavors. One is $x_0 < |m_f| < x_2$ and the other is $x_2 < |m_f| < \sqrt{\frac{3}{2}}x_2$. In the former case, since the position of flavor mass is on the branch cut $[ix_1, ix_2]$, $\mathcal{F}_{\rm flavor}$ can be written in terms of a dual period Π_1. Then by following the derivation in previous section[30] we obtain the matrix model curve near the singularity on the non-degenerated case as follows:

$$y_m^2 = x^2(x^2 - m_f^2)^2 + \langle f_2 \rangle x^2 + \langle f_0 \rangle = [x^2 \pm 2\eta\Lambda^2(1+c)]^2(x^2 \pm 4\eta\Lambda^2)\,,$$

where the fields, parameter of a superpotential, and glueball field are given by

$$\begin{aligned} \langle f_2 \rangle &= 4\eta^2\Lambda^4(1+2c)\,, \qquad \langle f_0 \rangle = \pm 16\eta^3\Lambda^6(1+c)^2\,, \\ m_f^2 &= \pm 2\eta\Lambda^2(2+c)\,, \qquad \langle S \rangle = -\eta^2\Lambda^4(1+2c)\,, \\ c &\equiv \cos\left[\frac{\pi(2\widetilde{N_1} - \widetilde{N_f})}{\widetilde{N} - \widetilde{N_f} - 2}\right], \qquad \eta^{\widetilde{N} - \widetilde{N_f} - 2} = i^{\widetilde{N_f}} \end{aligned} \tag{7.1}$$

together with (6.5). We can see that the solutions for the roots $x_2^2 = \pm 4\eta\Lambda^2$ and $x_0^2 = \pm 2\eta\Lambda^2(1+c)$ satisfy the relation $2x_0^2 \leq x_2^2$, implying that these solutions are consistent with the condition we have assumed before (Note $|m_f| = \sqrt{(2x_0^2 + x_2^2)/2} < x_2$). Note that c should be negative or zero but $c \neq -1$. When $c = -1$, then this will be the vacuum in the degenerated case. The constraint $|m_f| < x_2$ is too restrictive and most of the examples from factorization problem [19, 20] are beyond this criterion.

There exists one more possibility, $x_2 < |m_f| < \sqrt{\frac{3}{2}}x_2$. In this case, since $\mathcal{F}_{\rm flavor}$ cannot be written in terms of a dual period Π_1, we cannot obtain a general matrix model curve. Namely, the integral $\mathcal{F}_{\rm flavor}$ can be represented by an elementary function but when we solve the equations to obtain the roots x_0 and x_2, a general solution for the matrix model curve is not apparent.

In summary, at $n = 0$ and $n = 1$ singularity we obtain some partial results for matrix model curve (the curve does not exist for all the range of N, N_1, and N_f but for some particular values) in which the condition, c is negative or zero ($c \neq -1$), should be satisfied. To see the correctness of the result let us study an example satisfying this condition only.[31]

[30]Note that when we compute the equation of motion for f_0, since the extra piece from the flavor part, an integral $\int_{i\Lambda_0}^{\Lambda_0} \frac{\partial y_m}{\partial f_0} dx$, behaves like $\mathcal{O}(\Lambda_0^{-2})$, it does not contribute. Then the equation of motion for f_0 looks similar to the pure case [15] and it is given by $\frac{x_2^2 - 2x_0^2}{x_2^2} = -\cos\left[\frac{\pi(2\widetilde{N_1} - \widetilde{N_f})}{\widetilde{N} - \widetilde{N_f} - 2}\right]$ and the equation of motion for f_2 leads to $(\widetilde{N} - \widetilde{N_f} - 2)\log\left|\frac{x_2^2}{4\Lambda^2}\right| + N_f \log i = 0$. After solving this, one gets $x_2^2 = \pm 4\eta\Lambda^2$, and $\eta^{\widetilde{N} - \widetilde{N_f} - 2} = i^{\widetilde{N_f}}$.

[31]For SO(5) with $N_f = 2$ (non-baryonic $r = 0$ branch), the matrix model curve was given in [19] as $\widehat{y_m}^2 = \left(x^2 - m_f^2\right)^2\left(x^2 - 4\Lambda^2\right)$. To see whether there exists the equivalence of two approaches, we substitute $(N_0, N_1, N_f) = (3, 1, 2)$ into our general formula (7.1) and then obtain the matrix model curve: $y_m^2 = \left(x^2 \pm 2\eta\Lambda^2\right)^2\left(x^2 \pm 4\eta\Lambda^2\right)$ where $\eta = -1$ with $m_f^2 = \pm 4\eta\Lambda^2$. However, in this case, $c = 1$. Therefore, we cannot see an agreement between two approaches. That is, the matrix model curve exists only from the strong-coupling approach. One can easily see that for SO(6) with $N_f = 1$ (non-baryonic $r = 0$ branch), the c value is 1/2, for SO(6) with $N_f = 2$ (non-baryonic $r = 0$ branch), $c = 1$, and for SO(6) with $N_f = 3$ (non-baryonic $r = 0$ branch) the c value is -1. Most of the examples studied in [19] belong to the case where c is positive or -1.

SO(5) with $N_f = 1$: non-baryonic $r = 0$ branch. The matrix model curve derived by the factorization was given in [19] as (There is a typo in the presentation of [19]) $\widehat{y_m}^2 = x^2(x^2 - m_f^2)^2 - 4\Lambda^4(x^2 - m_f^2)$. This matrix model curve has an extra double root at $m_f^2 = 4\eta i\Lambda^2$ where $\eta^2 = 1$ and have the following factorized form, $\widehat{y_m}^2 = (x^2 \pm 2i\Lambda^2)^2(x^2 \pm 4i\Lambda^2)$. To see the equivalence of two approaches we substitute $(N_0, N_1, N_f) = (3, 1, 1)$ where $\mathrm{SO}(5) \to \mathrm{SO}(3) \times \widehat{\mathrm{U}(1)}$ and $N = N_0 + 2N_1$ into our general formula (7.1) and then obtain the matrix model curve:

$$y_m^2 = (x^2 \pm 2\eta\Lambda^2)^2(x^2 \pm 4\eta\Lambda^2)$$

where $\eta^3 = i$. Therefore for pure imaginary $\eta = -i$, we see the agreement.

For USp($2N$) case, as already discussed in section 4, we have to pay attention to the signs in the matrix model curve. To get the right results for USp($2N$) theory from the results for SO(N) theory, we simply replace $x_0^2 \to -x_0^2, x_2^2 \to -x_2^2$, and $(N-2) \to (2N+2)$ respectively. With this in mind, the matrix model can be written as

$$y_m^2 = x^2(x^2 - m_f^2)^2 + \langle f_2 \rangle x^2 + \langle f_0 \rangle = [x^2 \pm 2\eta\Lambda^2(1+c)]^2(x^2 \pm 4\eta\Lambda^2)\,,$$

where the fields, parameter of a superpotential, and glueball field are given by

$$\begin{aligned} \langle f_2 \rangle &= 4\eta^2\Lambda^4(1+2c)\,, \qquad \langle f_0 \rangle = \pm 16\eta^3\Lambda^6(1+c)^2\,, \\ m_f^2 &= \pm 2\eta\Lambda^2(2+c)\,, \qquad \langle S \rangle = -\eta^2\Lambda^4(1+2c)\,, \\ c &\equiv \cos\left[\frac{\pi(2\widetilde{N_1} - \widetilde{N_f})}{2\widetilde{N} - \widetilde{N_f} + 2}\right], \qquad \eta^{2\widetilde{N} - \widetilde{N_f} + 2} = i^{\widetilde{N_f}}\,, \end{aligned} \tag{7.2}$$

with the modified quantities are given by $2\widetilde{N} = 2N - 2r$, $\widetilde{N_f} = N_f - 2r$, and $2\widetilde{N_1} = 2N_1 - 2r$. In addition, we can see that the solutions for different parametrization are given by $x_2^2 = \pm 4\eta\Lambda^2$, and $x_0^2 = \pm 2\eta\Lambda^2(1+c)$ which satisfy the relation $2x_0^2 \leq x_2^2$, implying that these solutions are consistent with the condition we have assumed before. Note that c defined in (7.2) should be negative or zero ($c \neq -1$).[32] Let us consider one example.

USp(4) with $N_f = 2$: non-baryonic $r = 1$ branch. The matrix model curve can be written as $\widehat{y_m}^2 = t^3 - 2m_f^2 t^2 + \left(-4\Lambda^4 + m_f^4\right) t + 4\Lambda^4 m_f^2$ with $a = -2m_f^2$. Requiring an extra double root it becomes $\widehat{y_m}^2 = \left(t \pm 2i\Lambda^2\right)^2 \left(t \pm 4i\Lambda^2\right)$ at $m_f^2 = \mp 4i\Lambda^2$. To see the

[32] For USp(4) with $N_f = 2$ (non-baryonic $r = 1$ branch), the matrix model curve is $\widehat{y_m}^2 = t^3 + 2(m_f^2 - s_1)t^2 + (-4\Lambda^4 + m_f^4 - 2m_f^2 s_1 + s_1^2)t - 4(\Lambda^4 m_f^2 - \Lambda^4 s_1)$ with $a = -2m_f^2$. Requiring an extra double root, the matrix model curve becomes $\widehat{y_m}^2 = (t \pm 2i\Lambda^2)^2(t \pm 4i\Lambda^2)$ at $m_f^2 = \mp 4i\Lambda^2$. To see whether the equivalence arises, let us put $(\widetilde{N}, \widetilde{N_1}, \widetilde{N_f}) = (1, 0, 0)$ into (7.2) where the breaking pattern is $\mathrm{USp}(4) \to \widehat{\mathrm{U}(1)} \times \mathrm{USp}(2)$ we obtain the matrix model as $y_m^2 = \left(x^2 \pm 4\eta\Lambda^2\right)^3, \eta^4 = +1$. In this case, since $c = 1$, one cannot see the agreement between two approaches. In other words, there exists only a matrix model curve from strong-coupling approach. One can easily see that for USp(6) with $N_f = 2$ (non-baryonic $r = 0$ branch), the c value is 1/2, for USp(6) with $N_f = 4$ (non-baryonic $r = 0$ branch), $c = 1$, and for USp(6) with $N_f = 6$ (non-baryonic $r = 0$ branch) the c value is -1. Most of the examples studied in [20] belong to the case where c is positive or -1.

JHEP02(2004)009

equivalence we put the values $(2\widetilde{N}, 2\widetilde{N_1}, \widetilde{N_f}) = (2,2,0)$ where $\mathrm{USp}(4) \to \mathrm{USp}(0) \times \widehat{\mathrm{U}(2)}$ into (7.2) we obtain the matrix model

$$y_m^2 = (x^2 \pm 2\eta\Lambda^2)^2 (x^2 \pm 4\eta\Lambda^2)\,, \qquad \eta^4 = 1\,.$$

For the cases of pure imaginary $\eta = \pm i$, the matrix model curve from the glueball approach coincides with the curve from strong-coupling approach.

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